FROM ISOMETRIC EMBEDDINGS TO TURBULENCE

Contents

1. Introduction

The following paradox concerning isometric embeddings of the sphere S^2 is well-known: whereas the only C^2 isometric embedding of S^2 into $\mathbb{R}^3$ is the standard embedding modulo rigid motion, there exist many C^1 isometric embeddings which can *wrinkle* S^2 into arbitrarily small regions. The latter *flexibility* follows from the celebrated Nash-Kuiper theorem [Nas54, Kui55]. The proof involves an iteration scheme called convex integration which turned out to have surprisingly wide applicability.

More generally, this type of flexibility appears in a variety of different geometric contexts and is known as the *h-principle* [Gro86]. But one has to distinguish two contrasting cases. In problems which are formally (highly) under-determined, such as isometric embeddings into Euclidean space with high codimension, one might expect to find flexibility among smooth solutions. On the other hand in problems which are formally determined (or even in some cases over-determined), like embedding a surface into $\mathbb{R}^3$, the flexibility can only be expected at very low regularity. In fact this can be taken as a rule of thumb: the h-principle may appear in either high codimension or low regularity. In both cases new techniques are required because of the essential non-uniqueness. However, the techniques for proving the h-principle differ substantially in the two cases.

Quite surprisingly, the same ideas can be applied to a variety of equations from fluid dynamics. These examples obviously belong to the "low regularity" case. The purpose of these notes is to explain how suitable variants of convex integration can be used to construct very large sets of weak solutions to such equations, most prominently to the incompressible Euler equations.

After a brief survey, based on [DLS12b] of the available results concerning weak solutions, we present the original proof due to Nash of the celebrated Nash-Kuiper theorem. There are very good presentations already available in the literature, for instance [EM02]. However, contrary to geometry texts, our purpose is first to isolate the key ideas and transfer them from C^1 to a Lipschitz setting, where they can be applied to weak solutions of the Euler equations. In fact, these ideas can be applied in a very general framework, originally due to L. Tartar [Tar79], which consists of a plane-wave analysis in the phase space. This framework has been developed in the last 20 years to a very powerful theory, see [DM97, MŠ03]. We then show that with this framework at hand, the celebrated results of Scheffer and Shnirelman [Sch93, Shn97, Shn00] concerning the existence of weak solutions to the Euler equations with compact support in space-time, can be recovered [DLS09, DLS10].

Finally, we take another look at the Nash-Kuiper theorem and analyse whether the construction can be extended to produce more regular solutions [Bor65, Bor04, CDLS12]. The motivation for this comes from Onsager's theory of turbulence [Ons49], which predicts the existence of certain weak solutions of the Euler equations.

2. Non-uniqueness for the Euler equations

In this section we give a brief survey of what is known concerning weak solutions of the incompressible Euler equations. For simplicity we will restrict attention to periodic boundary conditions. A more complete survey can be found in [DLS12b]. The incompressible Euler equations can be written as

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v(0, \cdot) = v_0, \end{cases} \tag{1}$$

where the unknowns v and p are, respectively, a vector field and a scalar function defined on $\mathbb{T}^n \times [0, T)$, where $\mathbb{T}^n$ is the n-dimensional (flat) torus.

By a weak solution we mean, as usual, an L^2 vector field which solves the equations in the sense of distributions. In other words $v \in L^2(\mathbb{T}^n \times (0, T))$ is a weak solution of the incompressible Euler equations if

$$\int_0^T \int_{\mathbb{T}^n} \partial_t \varphi \cdot v + \nabla \varphi : v \otimes v \, dx dt = 0 \tag{2}$$

for all $\varphi \in C^\infty(\mathbb{T}^n \times (0, T); \mathbb{R}^n)$ with $\operatorname{div} \varphi = 0$ and

$$\int_0^T \int_{\mathbb{T}^n} v \cdot \nabla \psi \, dx dt = 0 \qquad \text{for all } \psi \in C^\infty(\mathbb{T}^n \times (0, T)). \tag{3}$$

When $v_0 \in L^2(\mathbb{T}^n)$, the vector field v is a weak solution of (1) if (2) can be replaced by

$$\int_0^T \int_{\mathbb{T}^n} \partial_t \varphi \cdot v + \nabla \varphi : v \otimes v \, dxdt + \int_{\mathbb{T}^n} \varphi(x,0) \cdot v_0(x)\, dx = 0 \tag{4}$$

for all $\varphi \in C^\infty(\mathbb{R}^n \times [0,T); \mathbb{R}^n)$ with $\operatorname{div} \varphi = 0$.

The first non-uniqueness result for weak solutions of (1) is due to Scheffer in his groundbreaking paper [Sch93]. The main theorem of [Sch93] states the existence of a nontrivial weak solution in $L^2(\mathbb{R}^2 \times \mathbb{R})$ with compact support in space and time. Later on Shnirelman in [Shn97] gave a different proof of the existence of a nontrivial weak solution in $L^2(\mathbb{T}^2 \times \mathbb{R})$ with compact support in time. In these constructions it is not clear if the solution belongs to the *energy space* $L^\infty(0,T; L^2(\mathbb{T}^n))$. In the paper [DLS09] a relatively simple proof of the following stronger statement was given:

Theorem 2.1 (Non-uniqueness of weak solutions)**.** *There exist infinitely many compactly supported weak solutions of the incompressible Euler equations in any space dimension. In particular there are infinitely many solutions $v \in L^\infty(0,T; L^2(\mathbb{T}^n))$ to (1) for $v_0 = 0$ and arbitrary $n \geq 2$.*

In fact, with similar techniques one can construct solutions to arbitrary initial data in the sense of (4), see [Wie11].

Theorem 2.2 (Global existence for weak solutions)**.** *Let $v_0 \in L^2(\mathbb{T}^n)$ be a solenoidal vectorfield. Then there exist infinitely many global weak solutions of (1) with bounded energy, i.e. such that*

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^n} |v(x,t)|^2 \, dx$$

is bounded. Moreover $E(t) \to 0$ as $t \to \infty$.

The weak solutions constructed in Theorems 2.1-2.2 are in general such that the kinetic energy $\int |v(x,t)|^2 \, dx$ has an instantaneous jump at time zero, in particular the energy is allowed to increase. On the other hand on physical grounds one might impose that the energy should be non-increasing. It is quite remarkable that this condition already singles out the unique classical solution if it exists [Lio96]:

Theorem 2.3 (Weak-strong uniqueness)**.** *Let $v \in L^\infty([0,T), L^2(\mathbb{T}^n))$ be a weak solution of (1) with the additional property that $\nabla v + \nabla v^T \in L^1([0,T), L^\infty(\mathbb{T}^n))$. Assume that $w \in L^\infty([0,T), L^2(\mathbb{T}^n))$ is another weak solution of (1) satisfying*

$$\int_{\mathbb{T}^n} |w(x,t)|^2 \, dx \;\leq\; \int_{\mathbb{T}^n} |v_0|^2(x)\, dx \qquad \text{for a.e. } t. \tag{5}$$

Then w coincides with v as long as the latter exists.

This theorem has recently been generalized to admissible measure-valued solutions in [BDLS11], leading to the observation that dissipative solutions of P.L. Lions are essentially the same as admissible measure-valued solutions.

In light of this weak-strong uniqueness, let us pause for a moment to discuss the issue of energy conservation. It is easy to see that C^1 solutions of the incompressible Euler equations satisfy the following identity, which expresses the conservation of the kinetic energy in local form:

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) = 0\,. \tag{6}$$

Integrating (6) in space we formally get the conservation of the total kinetic energy

$$\frac{d}{dt}\int_{\mathbb{T}^n} \frac{|v|^2}{2}(x,t)\,dx = 0\,. \tag{7}$$

For weak solutions, the energy conservation (7) might be violated, and indeed, this possibility has been considered for a rather long time in the context of 3 dimensional turbulence. In his famous note [Ons49] about statistical hydrodynamics, Onsager considered weak solutions satisfying the Hölder condition

$$|v(x,t) - v(x',t)| \le C|x - x'|^{\alpha}, \tag{8}$$

where the constant C is independent of $x, x' \in \mathbb{T}^3$ and t. He conjectured that

(a) Any weak solution v satisfying (8) with $\alpha > \frac{1}{3}$ conserves the energy;
(b) For any $\alpha < \frac{1}{3}$ there exist weak solutions v satisfying (8) which do not conserve the energy.

This conjecture is also very closely related to Kolmogorov's famous K41 theory [Kol91] for homogeneous isotropic turbulence in 3 dimensions. We refer the interested reader to [Fri95, Rob03, ES06]. Part (a) of the conjecture is by now fully resolved: it has first been considered by Eyink in [Eyi94] following Onsager's original calculations and proved by Constantin, E and Titi in [CET94]. Slightly weaker assumptions on v (in Besov spaces) were subsequently shown to be sufficient for energy conservation in [DR00a, CCFS08].

Concerning part (b) of the conjecture, therefore it is of interest to study the possibility that for sufficiently irregular weak solutions the energy is decreasing in time. In particular, this motivates studying admissible weak solutions.

The first example of a weak solution in the energy space for which the energy is a strictly decreasing function of time was produced by A. Shnirelman in [Shn00].

More generally, it turns out that one can construct weak solutions v with prescribed energy density $\frac{1}{2}|v|^2$ - in other words, for a given positive function $\bar{e}(x,t)$ we can find a weak solution v of the Euler equations such that $\frac{1}{2}|v|^2 = \bar{e}$. Let us delay stating the precise result until Section 6, and focus in this introduction on the following striking consequence: admissible weak solutions need not be unique. A particularly simple demonstration is furnished by the following example. Consider the following solenoidal vector field in $\mathbb{T}^2 = (-\pi,\pi)^2$:

$$v_0(x) = \begin{cases} (1,0) & \text{if } x_2 \in (-\pi,0) \\ (-1,0) & \text{if } x_2 \in (0,\pi) \end{cases} \tag{9}$$

and extended periodically. We have the following result from [Szé11]:

Theorem 2.4 (The vortex-sheet is wild)**.** *For v_0 as in (9) there are infinitely many weak solutions of (1) on $\mathbb{T}^2 \times [0,\infty)$ which satisfy (5).*

Any initial data v_0 which leads to non-uniqueness of admissible weak solutions is, a fortiori, irregular. Indeed, this follows from the weak-strong uniqueness Theorem 2.3 together with classical local existence results for regular initial data. We call initial data, for which admissible weak solutions are not unique, "wild". One might ask how large the set of these "wild" initial data is. It turns out that this is a dense set in L^2, see Theorem 2 in [SW12]:

Theorem 2.5 (Density of wild initial data)**.** *The set of wild initial data is dense in the space of L^2 solenoidal vectorfields.*

For a thorough discussion of further, more stringent "admissibility" criteria based on (6)-(7) we refer the reader to [DLS10, DLS12b].

Coming back to Onsager's conjecture, recently weak solutions in dimension $n = 3$ satisfying (8) were constructed in [DLS13, DLS12a], where the energy is strictly decreasing. More precisely,

Theorem 2.6 (Non-conservation of energy)**.** *Let $e : [0,1] \to \mathbb{R}$ be a smooth positive function. For every $\alpha < \frac{1}{10}$ there exists a weak solution $v \in C(\mathbb{T}^3 \times [0,1])$ such that (8) holds and*

$$e(t) = \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx \qquad \forall t \in [0,1]\,. \tag{10}$$

In fact the pressure also enjoys additional Hölder regularity, see [DLS12a] for details. A presentation of the proof of this result would go beyond the scope of these notes. We will show how to improve the Nash-Kuiper result to $C^{1,\alpha}$ in Section 7. Although several additional ideas are needed for the proof of Theorem 2.6, we hope that the reader will be at least convinced by the philosophy of these lecture notes, namely that the analogies between weak solutions of the Euler equations and rough isometric embeddings are far reaching, so that a proof of Theorem 2.6 should go along the lines of the more explicit proof in Section 7 for embeddings.

All of the results presented in this section rely on a general construction called convex integration. In the next couple of sections we will develop this theory in some detail, starting with a seemingly completely unrelated problem: the construction of isometric embeddings.

3. The Nash-Kuiper Theorem

The starting point for the story of convex integration is the following very surprising theorem.

Theorem 3.1 (Nash-Kuiper)**.** *Let (M^n, g) be a smooth compact manifold, $m \geq n+1$ and*

$$u : M^n \hookrightarrow \mathbb{R}^m$$

a short embedding. Then u can be uniformly approximated by C^1 isometric embeddings.

Recall that a short map is one which shrinks distances, in other words

$$\ell(u \circ \gamma) \leq \ell(\gamma)$$

for any C^1 curve $\gamma \subset M^n$, where ℓ denotes the length.

A way to "visualize" this theorem is to imagine wrinkling the standard n-sphere $S^n \subset \mathbb{R}^{n+1}$ inside a very tiny ball B_ε. Indeed, the map which homothetically shrinks $S^n \to \varepsilon S^n$ is clearly a short embedding. Therefore the theorem implies that in a C^0 neighbourhood of this shrinking map there exist C^1 isometric embeddings; in

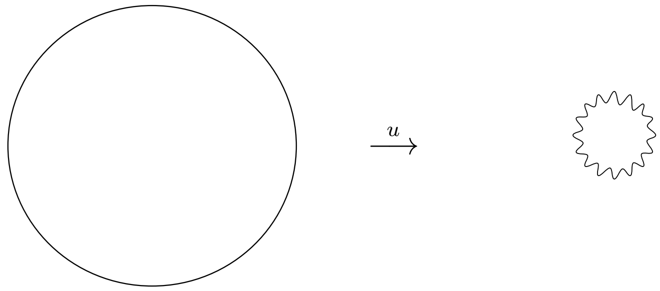

FIGURE 1. Nash-Kuiper in dimension $n = 1$

particular there exist C^1 isometric images of S^n in an arbitrarily small neighbourhood of εS^n. It is on the other hand important to note, that, since the isometry constructed is C^1, the process is very different from the "usual" crumbling of paper for instance. The latter produces folds, leading to a Lipschitz map, whereas the Nash-Kuiper theorem produces continuous tangents. In dimension $n = 1$ this is pretty trivial, see Figure 1, even with a smooth isometric embedding. On the other hand for dimension $n \geq 2$ the classical rigidity of the sphere [CV27, Her43] implies that any such isometry cannot be C^2. We will briefly return to the issue of rigidity in Section 7. For a comprehensive introduction to rigidity we refer to Chapter 12 in [Spi79].

3.1. **Local version in codimension 2.** In order to isolate the analytical ideas in the proof of the Nash-Kuiper theorem, it helps to first consider a local version. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary, which we can think of as a coordinate patch on M^n, and let g be a smooth metric on Ω. In other words, $g \in C^\infty(\overline{\Omega}; \mathcal{P})$, where

$$\mathcal{P} = \{n \times n \text{ positive definite matrices}\} .$$

Furthermore, let $m \geq n+1$. A map $u : \overline{\Omega} \to \mathbb{R}^m$ is an immersion if

$$\nabla u^T \nabla u = \left(\partial_i u \cdot \partial_j u\right)_{i,j=1\dots n}$$

is non-singular for every $x \in \overline{\Omega}$. Moreover, in this case $\nabla u^T \nabla u$ is the induced metric on the image $u(\Omega)$. Thus, the immersion is short if

$$\nabla u^T \nabla u \leq g \quad \text{in } \Omega$$

in the sense of quadratic forms, and it is isometric if

$$\nabla u^T \nabla u = g \quad \text{in } \Omega.$$

A smooth strictly short immersion is therefore an immersion $u \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ such that

$$\nabla u^T \nabla u < g \quad \text{in } \overline{\Omega} \tag{11}$$

Theorem 3.2. *Let $m \geq n+2$ and $u : \Omega \to \mathbb{R}^m$ a smooth strictly short immersion. For any $\varepsilon > 0$ there exists $\tilde{u} \in C^1(\overline{\Omega}; \mathbb{R}^m)$ such that $\|u - \tilde{u}\|_{C^0(\Omega)} < \varepsilon$ and*

$$\nabla \tilde{u}^T \nabla \tilde{u} = g \text{ in } \overline{\Omega}.$$

Before proceeding to the proof, let us show the main idea of Nash. Let u be a strictly short map as in Theorem 3.2. Then $u(\Omega) \subset \mathbb{R}^m$ is a smooth submanifold of codimension at least 2. Therefore we can choose two linearly independent unit normal vectors ζ, η to $u(\Omega)$. In other words, there exist $\zeta, \eta \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ such that for all $x \in \Omega$

$$|\zeta| = |\eta| = 1, \quad \zeta \cdot \eta = 0, \text{ and } \nabla u^T \zeta = \nabla u^T \eta = 0. \tag{12}$$

Next, let $\xi \in S^{n-1}$ a direction in $\mathbb{R}^n$ and set

$$v(x) = u(x) + \frac{a(x)}{\lambda}\Big(\sin(\lambda x \cdot \xi)\zeta(x) + \cos(\lambda x \cdot \xi)\eta(x)\Big) \tag{13}$$

for some amplitude a and frequency $\lambda \gg 1$. Then

$$\nabla v = \nabla u + a(x)\Big(\cos(\lambda x \cdot \xi)\zeta \otimes \xi - \sin(\lambda x \cdot \xi)\eta \otimes \xi\Big) + O\left(\frac{1}{\lambda}\right), \tag{14}$$

so that, because of (12)

$$\nabla v^T \nabla v = \nabla u^T \nabla u + a^2(x)\xi \otimes \xi + O\left(\frac{1}{\lambda}\right). \tag{15}$$

In other words, the spiral perturbation in (13) leads to a new map v, whose induced metric, given by (15) is – up to an error of size λ^{-1} – *increased* in the direction of ξ by an amount a^2 and is *not changed* in orthogonal directions. This is where the shortness assumption comes into play: since u is assumed to be strictly short, we can write

$$g - \nabla u^T \nabla u = \sum_k a_k^2 \xi^k \otimes \xi^k. \tag{16}$$

Then, by successively adding spirals as in (13) and choosing λ in each step sufficiently large, we should be able to *correct* the initial metric, up to an arbitrarily small error. Moreover, (14) implies that

$$\|\nabla v - \nabla u\|_{C^0(\Omega)} \sim \|a\|_{C^0(\Omega)} + O\left(\frac{1}{\lambda}\right) \tag{17}$$

whereas from (16) we obtain

$$\|a\|_{C^0(\Omega)} \leq \|g - \nabla u^T \nabla u\|_{C^0(\Omega)}^{1/2}.$$

In this way it is possible to control the C^1 norm of the perturbations.

There is, however, one important detail: the calculations just shown only work if the direction ξ in (13) is independent of x. Therefore in (16) one cannot simply diagonalize the positive definite matrix $g(x) - \nabla u(x)^T \nabla u(x)$ for each x. Instead, we define a kind of partition of unity on $\mathcal{P}$, the space of positive definite matrices.

Lemma 3.3 (Decomposing the metric error). *There exists a sequence $\{\xi^k\}$ of unit vectors in $\mathbb{R}^n$ and a sequence $\Gamma_k \in C_c^\infty(\mathcal{P}; [0, \infty))$ such that*

$$A = \sum_k \Gamma_k^2(A)\xi^k \otimes \xi^k \qquad \forall A \in \mathcal{P},$$

and there exists a number $N \in \mathbb{N}$ depending only on n such that, for all $A \in \mathcal{P}$ at most N of the $\Gamma_k(A)$ are nonzero.

Proof. First of all observe that $\mathcal{P}_1 := \mathcal{P} \cap \{\mathrm{tr}\,(A) = 1\}$ is an open convex subset of $L := \{A \in \mathbb{R}^{n\times n}_{sym} : \mathrm{tr}\,(A) = 1\}$, where $\dim L = \frac{n(n+1)}{2} - 1$. Therefore, by Caratheodory's theorem on convex sets, every element of $\mathcal{P}_1$ is contained in the interior of a simplex $\{A_1, \dots, A_{\frac{n(n+1)}{2}}\}^{co} \subset L$.

Let $\{S^{(i)}\}_{i\in\mathbb{N}}$ be a locally finite covering of $\mathcal{P}_1$ by non-degenerate open simplices in L. Thus, each S_i has the form

$$S^{(i)} = \mathrm{int}\left\{A^{(i)}_1, \dots, A^{(i)}_{\frac{n(n+1)}{2}}\right\}^{co}$$

for some $A^i_j \in \mathcal{P}_1$ and there exists $N_0 \in \mathbb{N}$ depending only on n such that each $A \in \mathcal{P}_1$ is contained in at most N_0 simplices. In each $S^{(i)}$ then there exist smooth functions $\mu_{i,j} : S^{(i)} \longrightarrow (0,1)$ such that

$$A = \sum_j \mu^2_{i,j}(A) A^{(i)}_j \text{ for all } A \in S^{(i)}.$$

Moreover, each $A^{(i)}_j$, being diagonalizable and positive definite, can be written as

$$A^{(i)}_j = (c^{(i)}_{j,1})^2 \xi^{(i)}_{j,1} \otimes \xi^{(i)}_{j,1} + \cdots + (c^{(i)}_{j,n})^2 \xi^{(i)}_{j,n} \otimes \xi^{(i)}_{j,n}$$

where $c^{(i)}_{j,k} \in \mathbb{R}$ and $\xi^{(i)}_{j,k} \in S^{n-1}$. Finally, let ψ_i be a partition of unity subordinate to S_i, i.e. such that

$$\mathrm{supp}\,\psi_i \subset\subset S_i,$$
$$\sum_i \psi^2_i = 1 \text{ in } \mathcal{P}_1.$$

Then

$$A = \sum_{i=1}^{\infty} \sum_{j=1}^{\frac{n(n+1)}{2}} \sum_{k=1}^{n} \left(\psi_i(A)\mu_{i,j}(A) c^{(i)}_{j,k}\right)^2 \xi^{(i)}_{j,k} \otimes \xi^{(i)}_{j,k}$$

gives a decomposition for $A \in \mathcal{P}_1$. For general $A \in \mathcal{P}$ this leads to the required decomposition

$$A = \sum_{i=1}^{\infty} \sum_{j=1}^{\frac{n(n+1)}{2}} \sum_{k=1}^{n} \mathrm{tr}\,(A) \left(\psi_i\left(\tfrac{1}{\mathrm{tr}\,(A)}A\right) \mu_{i,j}\left(\tfrac{1}{\mathrm{tr}\,(A)}A\right) c^{(i)}_{j,k}\right)^2 \xi^{(i)}_{j,k} \otimes \xi^{(i)}_{j,k}$$

with $N = \frac{1}{2}N_0 n^2(n+1)$.

$\square$

In the terminology of Nash [Nas54] a *stage* consists of decomposing the metric error into primitive metrics as in Lemma 3.3 and successively adding each primitive metric in *steps* using the spirals (13). Thus, the goal of a stage is the following:

Proposition 3.4 (Stage: reducing the metric error)**.** *Let $m \geq n+2$ and $u : \Omega \to \mathbb{R}^m$ a smooth strictly short immersion. For any $\varepsilon > 0$ there exists a smooth strictly short immersion $\tilde{u} : \Omega \to \mathbb{R}^m$ such that*

$$\|g - \nabla\tilde{u}^T\nabla\tilde{u}\|_{C^0(\Omega)} \leq \varepsilon \tag{18}$$

$$\|\nabla u - \nabla\tilde{u}\|_{C^0(\Omega)} \leq C\|g - \nabla u^T\nabla u\|^{1/2}_{C^0(\Omega)} \tag{19}$$

$$\|u - \tilde{u}\|_{C^0(\Omega)} \leq \varepsilon \tag{20}$$

Proof. Let $h = g - \nabla u^T \nabla u$, so that $h \in C^\infty(\overline{\Omega}; \mathcal{P})$. According to Lemma 3.3

$$h(x) = \sum_k a_k^2(x)\, \xi^k \otimes \xi^k, \tag{21}$$

where

$$a_k(x) := \Gamma_k(h(x)).$$

Observe that the sum in (21) is finite with, say, $M \in \mathbb{N}$ terms, since $h(\overline{\Omega})$ is a compact subset of $\mathcal{P}$ and the covering in Lemma 3.3 is locally finite. Moreover, for each $x \in \overline{\Omega}$ at most N of the coefficients $a_k(x)$ are nonzero.

Fix $0 < \delta < 1/2$ so that $h(x) \geq \delta\, \mathrm{Id}$ in Ω. We define successively maps

$$u_0 = u,\, u_1, u_2, u_3, \dots, u_M$$

as follows. Given u_k, let

$$u_{k+1} = u_k + (1-\delta)^{1/2} \frac{a_k}{\lambda_k} \Big(\sin(\lambda_k x \cdot \xi^k) \zeta_k(x) + \cos(\lambda_k x \cdot \xi^k) \eta_k(x) \Big),$$

where ζ_k, η_k are unit normal vector fields to $u_k(\Omega)$, i.e. such that

$$|\zeta_k| = |\eta_k| = 1, \quad \zeta_k \cdot \eta_k = 0, \text{ and } \nabla u_k^T \zeta_k = \nabla u_k^T \eta_k = 0,$$

and λ_k is sufficiently large so that

$$\big\| \nabla u_{k+1}^T \nabla u_{k+1} - \big(\nabla u_k^T \nabla u_k + (1-\delta) a_k^2 \xi^k \otimes \xi^k \big) \big\|_{C^0(\Omega)} \leq \frac{\delta^2}{2M}.$$

This is possible in view of (15). Observe that both the normal fields ζ_k, η_k and the choice of frequency λ_k depend on the map u_k. Then u_M satisfies

$$\big\| g - \nabla u_M^T \nabla u_M - \delta\, h \big\|_{C^0(\Omega)} \leq \delta^2/2\,,$$

hence

$$g - \nabla u_M^T \nabla u_M \geq \delta\, h - \tfrac{\delta^2}{2} \mathrm{Id} \geq \tfrac{\delta^2}{2} \mathrm{Id} > 0$$

for all $x \in \Omega$. Therefore u_M is strictly short and moreover

$$\big\| g - \nabla u_M^T \nabla u_M \big\|_{C^0(\Omega)} \leq \delta \big\| h \big\| + \delta^2/2\,. \tag{22}$$

Furthermore, from (14) we obtain, for λ_k sufficiently large,

$$|\nabla u_M(x) - \nabla u(x)| \leq \sum_k \big[|a_k(x)| + \delta \big] \tag{23}$$

for any $x \in \Omega$. On the other hand, from (21) we see – by taking the trace – that

$$\|a_k\|_{C^0(\Omega)} \leq \big\| g - \nabla u^T \nabla u \big\|_{C^0(\Omega)}^{1/2}$$

for each k. Therefore, since for each $x \in \Omega$ the sum in (23) contains at most N terms, we obtain

$$\|\nabla u_M - \nabla u\|_{C^0(\Omega)} \leq N \big\| g - \nabla u^T \nabla u \big\|_{C^0(\Omega)}^{1/2} + N\delta \tag{24}$$

Similarly, we obtain from (13)

$$\|u_M - u\|_{C^0(\Omega)} \leq \delta. \tag{25}$$

Thus, by choosing $\delta > 0$ sufficiently small, from (22), (24) and (25) we deduce (18)-(20) for $\tilde{u} = u_M$ with $C = 2N$. $\square$

Proof of Theorem ??
Let $\varepsilon_k \to 0$ be a sequence such that

$$\sum_k \varepsilon_k \le \varepsilon \text{ and } \sum_k \varepsilon_k^{1/2} < \infty.$$

Using Proposition 3.4 we obtain a sequence of smooth, strictly short maps $u_k \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ such that $u_0 = u$ and for $k \ge 1$

$$\begin{aligned} \|g - \nabla u_k^T \nabla u_k\|_{C^0(\Omega)} &\le \varepsilon_k \,, \\ \|\nabla u_{k+1} - \nabla u_k\|_{C^0(\Omega)} &\le C\varepsilon_k^{1/2} \,, \\ \|u_{k+1} - u_k\|_{C^0(\Omega)} &\le \varepsilon_{k+1} \,. \end{aligned}$$

Therefore u_k is a Cauchy sequence in C^1 and converges to a limit $\tilde{u} \in C^1(\overline{\Omega}; \mathbb{R}^m)$. It follows then, that $\tilde{u}$ satisfies

$$\nabla \tilde{u}^T \nabla \tilde{u} = g \text{ in } \Omega$$

$$\|u - \tilde{u}\|_{C^0(\Omega)} \le \sum_k \varepsilon_k \le \varepsilon$$

This completes the proof.

3.2. **Extensions.** In this section we describe how to modify the proofs from Section 3.1 in order to prove the general statement of Theorem 3.1. This involves modifications in the following directions:

- from $m = n + 2$ to $m = n + 1$;
- from single charts to general manifolds;
- from immersions to embeddings.

The case $m = n + 1$.

Recall that the building block in the construction for the codimension 2 case was the spiral in (13). In codimension 1 we need to replace this by corrugations (called *strains* in [Kui55]). More precisely, let

$$\gamma : \Omega \times S^1 \to \mathbb{R}^2; \quad (x, t) \mapsto (\gamma_1, \gamma_2)$$

be a family of closed curves (i.e. 2π-periodic in t), parametrized by $x \in \Omega$, and set

$$v(x) = u(x) + \frac{1}{\lambda}\Big(\gamma_1(x, \lambda x \cdot \xi)\zeta(x) + \gamma_2(x, \lambda x \cdot \xi)\eta(x)\Big), \tag{26}$$

where η is the (unique) normal vector to $u(\Omega)$ as before, and ζ is still to be chosen. The new induced metric is then

$$\nabla v^T \nabla v = \nabla u^T \nabla u + \dot{\gamma}_1\big(\nabla u^T \zeta \otimes \xi + \xi \otimes \nabla u^T \zeta\big) + \big(\dot{\gamma}_1^2 |\zeta|^2 + \dot{\gamma}_2^2\big)\xi \otimes \xi + O\left(\frac{1}{\lambda}\right),$$

where $\dot{\gamma}$ denotes the derivative with respect to t. Thus, the natural choice for ζ is such that $\nabla u^T \zeta = \xi$, i.e.

$$\zeta = \nabla u(\nabla u^T \nabla u)^{-1}\xi,$$

so that the metric change becomes $(2\dot{\gamma}_1 + |\zeta|^2\dot{\gamma}_1^2 + \dot{\gamma}_2^2)\xi \otimes \xi$. A slightly more clever choice of vectors can lead to a more symmetric form: set

$$\tilde{\zeta} = \frac{\zeta}{|\zeta|^2}, \quad \tilde{\eta} = \frac{\eta}{|\zeta|},$$

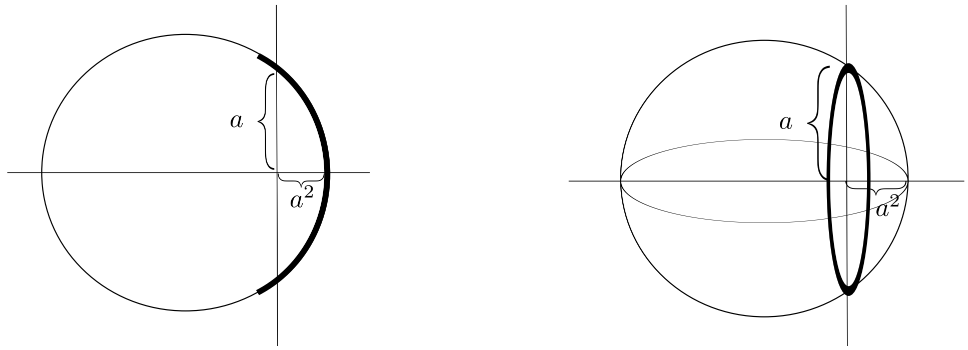

FIGURE 2. "Convex Integration": construct $\dot\gamma$ first with average zero, then integrate. The figure shows $\dot\gamma$ for the codimension 1 left and codimension 2 right.

and replace ζ, η by $\tilde\zeta, \tilde\eta$ in (26). We obtain

$$\nabla v^T \nabla v = \nabla u^T \nabla u + \frac{1}{|\zeta|^2}\left(2\dot\gamma_1 + \dot\gamma_1^2 + \dot\gamma_2^2\right)\xi \otimes \xi + O\left(\frac{1}{\lambda}\right). \tag{27}$$

Hence, in order to recover (15) we need to choose γ so that

(i) $(1+\dot\gamma_1)^2 + \dot\gamma_2^2 = |\zeta|^2 a^2 + 1$;
(ii) $t \mapsto \gamma(x,t)$ is 2π-periodic.

Observe that (i) should not be viewed a differential equation, since for any fixed x we can directly solve for $\dot\gamma$ and integrate in t, provided we replace (ii) by

(ii') $t \mapsto \dot\gamma(x,t)$ is 2π-periodic with average 0.

Thus, $\dot\gamma$ is required to solve an inclusion (i.e. take values on a circle) with average zero. In particular the origin needs to lie in the *convex hull* of the values of $\dot\gamma$.

Note that along the iteration of stages, the amplitude a will be small whereas $|\zeta|$ will stay order one (c.f. Proposition 3.4). Therefore we may write $\sqrt{1+|\zeta|^2a^2} \sim 1 + a^2$, see Figure 2 left. If we choose $\dot\gamma$ to take values only in the thickened part of the circle, we can ensure $|\dot\gamma| \le C|a|$, which leads to the C^1-estimate as in (17). The rest of the proof is now precisely as in the codimension 2 case.

General manifolds.

Fix a covering of the manifold M by coordinate charts

$$M \subset \bigcup_p U_p$$

with an associated partition of unity $\{\phi_p\}$ so that $\sum_p \phi_p = 1$ and $\phi_p \in C_c^\infty(U_p)$. Given a map $u : M \to \mathbb{R}^m$ let $u^\sharp e$ be the pullback of the standard Euclidean metric through u. At each *stage*, i.e. in the analogue of Proposition 3.4, we decompose the metric error

$$h = g - u^\sharp e$$

into primitive metrics in the different charts as

$$h(x) = \sum_{k,p} \phi_p(x) a_k^2(x) \xi^k \otimes \xi^k,$$

where $a_k(x) = \Gamma_k(h(x))$ as before. The proof of Proposition 3.4 can now be carried out with this decomposition in place of (21).

Embeddings.

In order to obtain embeddings, we need to ensure at each step that the perturbation (13) or (26) does not lead to self-intersections. To this end assume that we start with a smooth strictly short *embedding* u. Recall from (15) that the perturbation is chosen in such a way that

$$\|v-u\|_{C^0} \to 0 \text{ with } \lambda \to \infty, \tag{28}$$

and

$$\nabla v^T \nabla v = \nabla u^T \nabla u + a^2 \xi \otimes \xi + O\left(\frac{1}{\lambda}\right). \tag{29}$$

Let $x, y \in M$ be two points sufficiently close (in particular contained in a single chart). By the mean value theorem we find z on the line segment connecting x and y such that

$$v(y) - v(x) = \nabla v(z)(y-x),$$

and consequently

$$\begin{aligned}|v(y)-v(x)|^2 &= \big\langle \nabla v(z)^T \nabla v(z)(y-x), (y-x)\big\rangle \\ &\overset{(29)}{\geq} \big\langle \nabla u(z)^T \nabla u(z)(y-x), (y-x)\big\rangle + O\left(\tfrac{1}{\lambda}|x-y|^2\right) \\ &\geq |u(x)-u(y)|^2 \big(1 + o(|x-y|) + O(\tfrac{1}{\lambda})\big),\end{aligned}$$

where we have used that ∇u is continuous. Observe that, since along the iteration the gradients converge uniformly, the estimate above is *uniform* along the iteration.

Therefore there exists $\varepsilon > 0$ and $\lambda_0 > 1$ so that for all $\lambda \geq \lambda_0$

$$|v(y) - v(x)| \geq \frac{1}{2}|u(y) - u(x)| \text{ whenever } |x-y| < \varepsilon.$$

This ensures that no self-intersections are created locally. On the other hand, global self-intersections can be prevented using the C^0-control: given $\varepsilon > 0$ (28) implies the convergence

$$\frac{|v(y)-v(x)|}{|u(y)-u(x)|} \to 1 \text{ as } \lambda \to \infty$$

uniformly in the set $\{(x,y) \in M \times M : |x-y| \geq \varepsilon\}$. Therefore there exists $\lambda_1 > 1$ so that for all $\lambda \geq \lambda_1$

$$|v(y) - v(x)| \geq \frac{1}{2}|u(y) - u(x)| \text{ whenever } |x-y| \geq \varepsilon.$$

In conclusion, for sufficiently large λ the new map v is also an embedding.

3.3. **The Lipschitz case.** In the equidimensional case C^1 isometries do not enjoy the flexibility of Theorem 3.1. Indeed, it is very easy to see that any C^1 isometric map $\Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is necessarily locally affine (Liouville's theorem). One can consider Lipschitz maps instead. However, this leads to (at least) two different problem descriptions.

(A) $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz with

$$\nabla u(x)^T \nabla u(x) = g(x) \quad \text{ a.e. } x;$$

(B) the length of all rectifiable curves Γ is preserved in the sense that

$$\int_0^l \sqrt{g(\dot\Gamma(t),\dot\Gamma(t))}\,dt = \int_0^l |\nabla u(\Gamma(t))\dot\Gamma(t)|\,dt.$$

It is not difficult to see that $(B) \Rightarrow (A)$. Conversely, there exist maps satisfying (A) which do not satisfy (B). We will see this later in Section 5.1.

Although (B) is clearly a geometrically more "correct" notion of isometry, we will concentrate on maps satisfying (A). The reason for this is that the type of map satisfying (A) can be seen as the analogue of L^∞ weak solutions of the Euler equations as seen in Section 2.

Theorem 3.5. *Let $u:\Omega\to\mathbb{R}^n$ be a smooth strictly short map. For any $\epsilon>0$ there exists a Lipschitz map $\tilde u:\Omega\to\mathbb{R}^n$ such that $\|u-\tilde u\|_{C^0(\Omega)}<\epsilon$ and*

$$\nabla u(x)^T\nabla u(x) = g(x) \ \ a.e.\ x\in\Omega.$$

Although this theorem is essentially trivial (think of crumbling a piece of paper!) compared to Theorem 3.1, it is instructive to look at the analogues of the estimates in Proposition 3.4.

Proposition 3.6 (Stage in the Lipschitz case)**.** *Let $u:\Omega\to\mathbb{R}^n$ be a smooth strictly short map. For any $\varepsilon>0$ there exists a smooth strictly short map $\tilde u:\Omega\to\mathbb{R}^n$ such that*

$$\int_\Omega \mathrm{tr}(g-\nabla\tilde u^T\nabla\tilde u)\,dx \le \varepsilon \tag{30}$$

$$\int_\Omega |\nabla u-\nabla\tilde u|^2\,dx \le C\int_\Omega \mathrm{tr}(g-\nabla u^T\nabla u)\,dx \tag{31}$$

$$\|u-\tilde u\|_{C^0(\Omega)}\le\varepsilon \tag{32}$$

Actually, (31) essentially follows from the shortness (11) and the uniform estimate (32). To see this, let us write

$$\mathrm{tr}(g-\nabla\tilde u^T\nabla\tilde u) = \mathrm{tr}(g-\nabla u^T\nabla u) - 2\langle\nabla u,\nabla\tilde u-\nabla u\rangle - |\nabla\tilde u-\nabla u|^2.$$

Integrating by parts over Ω we obtain

$$\int_\Omega\langle\nabla u,\nabla\tilde u-\nabla u\rangle\,dx = -\int_\Omega \Delta u\cdot(\tilde u-u)\,dx + \int_{\partial\Omega}\langle Du,(\tilde u-u)\otimes\nu\rangle$$

and hence, using that $\tilde u$ is short, we deduce

$$\int_\Omega|\nabla\tilde u-\nabla u|^2\,dx \le \int_\Omega \mathrm{tr}(g-\nabla u^T\nabla u) + C\|u\|_{C^2(\Omega)}\|\tilde u-u\|_{C^0(\Omega)}.$$

Therefore, choosing $\varepsilon>0$ sufficiently small in (32) we can conclude (31).

The upshot is that in the statement of Proposition 3.6 we are allowed to *control* the C^0-norm of the perturbation $\tilde u-u$ whilst retaining the (strict) shortness condition (11), which provides a *uniform* gradient bound. This controlled uniform convergence then leads to strong convergence of the gradient, c.f.[MŠ03].

Sketch proof of Proposition 3.6. The proof proceeds analogously to the proof of Proposition 3.4, except now the metric error is measured in the L^1-norm rather than the C^0 norm. In particular, we start by decomposing the metric error into primitive metrics, and define successively maps in order to add each primitive metric.

For a single step we now consider a perturbation of the form

$$\tilde{u}(x) = u(x) + \frac{1}{\lambda}\gamma(x, \lambda x \cdot \xi)\tilde{\zeta}(x),$$

where, similarly to Section 3.2, $\gamma : \Omega \times S^1 \to \mathbb{R}$, and

$$\tilde{\zeta} = \frac{\zeta}{|\zeta|^2}, \quad \zeta = \nabla u(\nabla u^T \nabla u)^{-1}\xi.$$

The induced metric will then be

$$\nabla\tilde{u}^T\nabla\tilde{u} = \nabla u^T\nabla u + \frac{1}{|\zeta|^2}(2\dot{\gamma} + \dot{\gamma}^2)\xi\otimes\xi + O(\frac{1}{\lambda}).$$

In order to achieve the desired metric perturbation, we would require

(i) $(1 + \dot{\gamma})^2 = 1 + |\zeta|^2 a^2$;

(ii') $t \mapsto \dot{\gamma}(x, t)$ 2π-periodic with average zero.

However, the error estimate $O(\frac{1}{\lambda})$ in the metric change is only valid if $\gamma \in C^1(\Omega \times S^1)$. Therefore we have to replace (i) by a pointwise upper bound

$$(1 + \dot{\gamma})^2 \leq 1 + |\zeta|^2 a^2 \quad \forall\, x \in \Omega,\, t \in S^1$$

together with an *average* lower bound

$$\frac{1}{2\pi}\int_0^{2\pi}\left[a^2 - \frac{1}{|\zeta|^2}(2\dot{\gamma} + \dot{\gamma}^2)\right]dt \leq \varepsilon \quad \forall\, x \in \Omega.$$

The upper bound leads to the shortness condition (11). The estimate (30) follows from the average lower bound and the assertion, that for any $f \in C(\Omega \times S^1)$

$$\int_\Omega f(x, \lambda x\cdot\xi)\,dx \to \int_\Omega \frac{1}{2\pi}\int_0^{2\pi} f(x,t)\,dt\,dx$$

as $\lambda \to \infty$, applied to $f = \frac{1}{|\zeta|^2}(2\dot{\gamma} + \dot{\gamma}^2)$.

The rest of the proof is exactly as the proof of Proposition 3.4. □

We presented here a proof that follows the strategy of Nash. However, for Lipschitz differential inclusions a much more flexible and general technique is available. We will discuss this in the next section.

4. Convergence Strategies

In this chapter we discuss several ways of producing strongly convergent approximating sequences to differential inclusions from weakly convergent ones.

To motivate, let us revisit the Lipschitz version of the Nash theorem, Theorem 3.5. Let $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ be a bounded Lipschitz domain and let $\Gamma \subset \overline{\Omega}$ be a closed subset with $|\Gamma| = 0$ (for instance $\Gamma = \partial\Omega$). Furthermore, let $g \in C^\infty(\overline{\Omega}; \mathcal{P})$ be a smooth metric on Ω as before. Let

$$\begin{aligned} X_0 &= \left\{u \in C^\infty(\overline{\Omega}) : \nabla u^T\nabla u < g \text{ in } \overline{\Omega} \text{ and } u|_\Gamma = 0\right\} \\ X &= \text{closure of } X_0 \text{ in sup-norm} \end{aligned}$$

and

$$I(u) = \int_\Omega \operatorname{tr}(g - \nabla u^T\nabla u)\,dx.$$

Observe that $\|\nabla u\|_{C^0(\Omega)} \leq \|g\|^{1/2}_{C^0(\Omega)}$ for all $u \in X_0$, so that X consists of uniformly Lipschitz functions. Moreover, with the uniform topology X is a complete (in fact compact) metric space. Note that I is therefore well-defined on X, it is non-negative, but is not continuous (in fact it is upper-semicontinuous). The zero-set $\{u \in X : I(u) = 0\}$ consists of Lipschitz mappings such that

$$\begin{cases} \nabla u^T \nabla u = g & \text{a.e. } x \in \Omega, \\ u(x) = 0 & \forall\, x \in \Gamma. \end{cases} \tag{33}$$

Proposition 3.6, applied to the open set $\Omega \setminus \Gamma$, can be restated as follows:

$$\forall u \in X_0 \quad \exists u_k \in X_0 \text{ such that } u_k \to u \text{ uniformly in } \Omega \text{ and } I(u_k) \to 0.$$

The goal of this section is to give general statements that allow us to deduce from here

Theorem 4.1. *The set $\{u \in X : I(u) = 0\}$ is Baire-generic in X.*

Here, *Baire-generic* means residual. Recall that in a metric space a set is said to be *nowhere dense* if its closure has empty interior. A residual set is then the complement is a countable union of nowhere dense sets. We refer to [Oxt80] for a general reference on Baire category.

In particular Theorem 4.1 implies that $\{u \in X : I(u) = 0\}$ is dense in X. This shows that there exists a very large set of solutions of the problem (33). This statement should be compared with the density statement in the Nash-Kuiper theorem, Theorem 3.1.

The proof of Theorem 4.1 can be cast into a general framework for obtaining strongly convergent sequences from weakly convergent ones. Indeed, assuming Γ is nonempty, we see that the uniform topology in X is equivalent to the topology induced by weak convergence of ∇u. Writing $z = \nabla u$, we could directly consider sequences z_k with the constraint curl $z_k = 0$. We now consider the following "unconstrained" setting: let $\mathscr{D} \subset \mathbb{R}^d$ be an open bounded set and

$$X_0 \subset L^2(\mathscr{D}) \text{ bounded, } I : X_0 \to \mathbb{R} \text{ a functional}$$

with the property such that

$$\forall u \in X_0 \quad \exists u_k \in X_0 \text{ such that } u_k \rightharpoonup u \text{ in } L^2 \text{ and } I(u_k) \to 0. \tag{34}$$

Note that no continuity property is assumed on I. To have a simple but concrete setting in mind, consider the following example:

Example 1. *Consider the set*

$$X_0 := \left\{ u \in L^\infty(0,1) : |u(x)| < 1 \textit{ almost everywhere } \right\}$$

and let $I(u) = \int_0^1 1 - |u(x)|^2 dx$. Given $u \in X_0$ consider the functions

$$u_k(x) = u(x) + \frac{1}{2}(1 - |u(x)|^2)\sin(kx).$$

It is easy to see that $u_k \in X_0$ for all k. Moreover, with $f_k(x) := sin(kx)$

$$f_k \overset{*}{\rightharpoonup} 0 \textit{ and } f_k^2 \overset{*}{\rightharpoonup} 1/2 \textit{ in } L^\infty(0,1),$$

hence

$$u_k \overset{*}{\rightharpoonup} u \textit{ in } L^\infty(0,1) \textit{ and } \limsup_{k\to\infty} I(u_k) \le I(u) - \frac{1}{8} I(u)^2.$$

Consider now a sequence $\{u_k\} \subset X_0$ in the above example, defined inductively as $u_0 = 0$,

$$u_{k+1}(x) = u_k(x) + \frac{1}{2}(1 - |u_k(x)|^2)\sin(\lambda_{k+1}x),$$

for some sequence of frequencies $\{\lambda_k\}$. It is not difficult to see that if $\lambda_k \to \infty$ sufficiently fast, then the sequence $\{u_k\}$ converges strongly in $L^2(0,1)$ to some limit v with $|v| = 1$ a.e. The aim of this chapter is to show that this "choice of frequencies" can be adapted to achieve strong convergence in the general setting.

4.1. **Controlled weak convergence implies strong convergence.** Let $X_0 \subset L^2(\mathscr{D})$ be a bounded set and let X be the weak closure of X_0 (i.e. the closure with respect to the weak L^2 topology). Being a bounded subset of L^2, the weak topology on X is metrizable, making X into a compact metric space, which we will denote by (X,d).

Theorem 4.2. *Let $I : X \to \mathbb{R}$ be a functional on X which is continuous with respect to the strong topology. Assume that*

$$\forall\, u \in X_0 \quad \exists\, u_k \in X_0 \text{ with } u_k \rightharpoonup u \text{ in } L^2(\mathscr{D}),\ I(u_k) \to 0.$$

Then $\{u \in X : I(u) = 0\}$ is dense in (X,d).

Proof. Let $u \in X_0$. It suffices to prove that for any $\delta > 0$ there exists $w \in X$ with $I(w) = 0$ and $d(w,u) \leq \delta$.

To this end we construct a sequence $\{v_k\} \subset X_0$ inductively as follows. First of all, set $v_0 = u$. Having defined $v_1, \dots, v_k$, choose v_{k+1} so that

$$|\langle v_{k+1} - v_k, v_l\rangle| \leq 2^{-k} \quad \text{for all } l \leq k, \tag{35}$$

$$I(v_{k+1}) \leq 2^{-k}, \tag{36}$$

$$d(v_{k+1}, v_k) \leq \delta 2^{-k}\,, \tag{37}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Indeed, this is possible, since, by assumption, given v_k there exists a sequence $v_{k,j} \in X_0$ such that $v_{k,j} \rightharpoonup v_k$ and $I(v_{k,j}) \to 0$ as $j \to \infty$, and consequently also $|\langle v_{k,j} - v_k, v_l\rangle| \to 0$ as $j \to \infty$ for all $l \leq k$.

Using (35) we deduce that for any $m > n$

$$|\langle v_m - v_n, v_n\rangle| \leq \sum_{k=n}^{m-1} 2^{-k} \leq 2^{-n+1}.$$

Moreover, since X is bounded, we may extract a subsequence v_{n_j} such that $\|v_{n_j}\|_2 \to \alpha$ as $j \to \infty$ for some limit α. For this subsequence we have

$$\Big| \|v_{n_k} - v_{n_j}\|_2^2 - (\|v_{n_k}\|_2^2 - \|v_{n_j}\|_2^2) \Big| \leq 2|\langle v_{n_k} - v_{n_j}, v_{n_j}\rangle| \leq 2^{2-n_j} \quad \text{for all } k \geq j.$$

Consequently the sequence v_{n_j} is a Cauchy sequence and hence $v_{n_j} \to w$ strongly in L^2 for some $w \in X$. But then also $I(w) = 0$ by (36) and $d(w,u) \leq \delta$ by (37), as required. □

As an alternative to orthogonality in L^2, one may use mollifications in L^p. This argument is based on [MŠ03]. The setting here is the following: Let $X_0 \subset L^p(\mathscr{D})$ for some $1 < p < \infty$ be a bounded subset and X the weak closure. As before, X with the weak topology is a metric space (X,d).

Lemma 4.3. *Let $\rho \in C_c^\infty(\mathbb{R}^n)$. If $u_k \rightharpoonup u$ in $L^p(\mathbb{R}^n)$ for some $p < \infty$, then*

$$\rho * u_k \to \rho * u \text{ in } L^p_{loc}(\mathbb{R}^n).$$

Proof. For any $x \in \mathbb{R}^n$ the function $y \mapsto \rho(x-y)$ is in $L^{p'}(\mathbb{R}^n)$ so that

$$\rho * u_k(x) = \int \rho(x-y)u_k(y)dy \to \int \rho(x-y)u(y)dy.$$

Choose $q > p$ and let $r \geq 1$ be such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. By Young's inequality

$$\|\nabla(\rho * u_k)\|_q \leq C\|\nabla \rho\|_r \|u_k\|_p,$$

hence $\{\rho * u_k\}$ is bounded in $W^{1,q}(\mathbb{R}^n)$. Consequently, by Rellich's theorem combined with the pointwise convergence, for any bounded subset $\Omega \subset \mathbb{R}^n$ we have that $\rho * u_k \to \rho * u$ in $L^p(\Omega)$. □

Theorem 4.4. *Let $I : X \to \mathbb{R}$ be a functional on X which is continuous with respect to the strong L^p topology. Assume that*

$$\forall\, u \in X_0 \quad \exists\, u_k \in X_0 \text{ with } u_k \rightharpoonup u \text{ in } L^p(\mathscr{D}),\ I(u_k) \to 0.$$

Then $\{u \in X : I(u) = 0\}$ is dense in (X, d).

Proof. Let $\rho_\ell \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier kernel, so that $\rho \in C_c^\infty(\mathbb{R}^n)$, $\rho \geq 0$, $\int \rho = 1$ and $\rho_\ell(x) = \ell^n \rho(\ell^{-1}x)$. Given $u \in L^p(\mathscr{D})$ we define $\rho_\ell * u$ in the usual way by setting $u = 0$ outside $\mathscr{D}$.

Let $u \in X_0$. As before, it suffices to prove that for any $\delta > 0$ there exists $w \in X$ with $I(w) = 0$ and $d(w, u) \leq \delta$. We construct a sequence $\{v_k\} \subset X_0$ and a sequence of "scales" $\{\ell_k\}$ such that $\ell_k \to 0$, $I(v_k) \to 0$ and

$$\|v_k - \rho_{\ell_k} * v_k\|_p \leq 2^{-k} \tag{38}$$

$$\|\rho_{\ell_j} * (v_{k+1} - v_k)\|_p \leq 2^{-k} \text{ for all } j \leq k, \tag{39}$$

$$I(v_k) \leq 2^{-k}, \tag{40}$$

$$d(v_{k+1}, d_k) \leq \delta 2^{-k}. \tag{41}$$

Observe that here (38)-(39) take the role of the almost orthogonality (35). To see that this can be done, start with $v_0 = u$ and let $\ell_0 < 1$ be such that (38) with $k = 0$ holds. Having defined (v_j, ℓ_j) for $j = 1, \ldots, k$ satisfying (38) observe that (39) involves a finite number of inequalities to be satisfied by v_{k+1} and by Lemma 4.3 each one can be made arbitrarily small.

Next, we may assume without loss of generality that $v_k \rightharpoonup w$ in $L^p(\mathscr{D})$. Then

$$\begin{aligned}
\|v_k - w\|_p &\leq \|v_k - \rho_{\ell_k} * v_k\|_p + \|\rho_{\ell_k} * (v_k - w)\|_p + \|w - \rho_{\ell_k} * w\|_p \\
&\leq \|v_k - \rho_{\ell_k} * v_k\|_p + \sum_{j=k}^\infty \|\rho_{\ell_k} * (v_j - v_{j+1})\|_p + \|w - \rho_{\ell_k} * w\|_p \\
&\leq 2^{-k} + \sum_{j=k}^\infty 2^{-j} + \|w - \rho_{\ell_k} * w\|_p \\
&\leq 2^{-k} + 2^{-k+1} + \|w - \rho_{\ell_k} * w\|_p \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Moreover, as in Theorem 4.2, $I(w) = 0$ and $d(w, u) \leq \delta$. □

4.2. **Stability - the Baire category method.** Let us return to the L^2 setting (only for simplicity), so that X_0 is a bounded subset of $L^2(\mathscr{D})$ and X is the closure of X_0 in the weak topology. Consider

$$J(u) = \int_{\mathscr{D}} |u|^2 dx.$$

Obviously, in general $u_k \rightharpoonup u$ doesn't imply $J(u_k) \to J(u)$, so that J is not continuous on X. On the other hand J can be approximated pointwise by continuous maps. Indeed, as shown in Lemma 4.3

$$J_\varepsilon(u) = \int_{\mathscr{D}} |\rho_\varepsilon * u|^2 dx$$

is continuous with respect to weak convergence, and on the other hand $J_\varepsilon(u) \to J(u)$ as $\varepsilon \to 0$ for all $u \in L^2(\mathscr{D})$.

Definition 4.5. *In a metric space X a function $J : X \to \mathbb{R}$ is of class* Baire 1 *if it is a pointwise limit of continuous functions, i.e. if there exist $J_k \in C(X)$ such that $J_k(u) \to J(u)$ as $k \to \infty$ for all $u \in X$.*

The following theorem is a standard result in functional analysis.

Theorem 4.6. *If $J : X \to \mathbb{R}$ is a Baire-1 function on a complete metric space X, then the set of continuity points of J is a dense set in X.*

Proof. Let

$$E_{n,k} := \bigcap_{i,j \geq k} \big\{ u \in X : |J_i(u) - J_j(u)| \leq 1/n \big\}.$$

Since J_i is continuous, the set $E_{n,k}$ is closed for each n, k. Since $J_i(u) \to J(u)$ for all u,

$$X = \bigcup_{k=1}^{\infty} E_{n,k}.$$

In particular, by Baire's theorem the set

$$V_n := \bigcup_{k=1}^{\infty} \text{int } E_{n,k}$$

is open and dense. To see that it is dense let $B \subset X$ be open. Then $\overline{B}$ is - being a closed subset of X - itself a complete metric space, and $\bigcup_{k=1}^{\infty} \big(E_{n,k} \cap \overline{B}\big) = \overline{B}$. Therefore necessarily $\overline{B} \cap E_{n,k}$ has nonempty interior for some k, which in turn implies that $B \cap \text{int } E_{n,k} \neq \emptyset$, so that $B \cap V_n \neq \emptyset$.

But then the set $S = \bigcap_{n=1}^{\infty} V_n$ is dense. To conclude we prove that S consists of continuity points of J. Let $u \in S$ and $\varepsilon > 0$. Choose n so that $1/n < \varepsilon/3$. Then $u \in V_n$ and hence there exists $\delta_1 > 0$ and k such that $B_{\delta_1}(u) \subset E_{n,k}$. For any $v \in B_{\delta_1}(u)$

$$|J_i(v) - J_j(v)| < \varepsilon/3 \text{ for all } i, j \geq k,$$

and in particular - by letting $i \to \infty$ - also $|J(v) - J_j(v)| < \varepsilon/3$ for all $j \geq k$. Also, there exists $\delta_2 > 0$ such that $|J_k(u) - J_k(v)| < \varepsilon/3$ for all $v \in B_{\delta_1}(u)$. Hence, with $\delta = \min\{\delta_1, \delta_2\}$

$$|J(u) - J(v)| \leq |J(u) - J_k(u)| + |J_k(u) - J_k(v)| + |J_k(v) - J(v)| < \varepsilon$$

for all $v \in B_\delta(u)$. This proves that u is a point of continuity of J. $\square$

Let

$$S := \{u \in X : u \text{ is a continuity point of } J\}.$$

Observe that if $u \in S$, then

$$\forall\, u_k \in X \text{ with } u_k \rightharpoonup u \text{ we have } u_k \to u.$$

In light of this the set S is called the set of *stable* elements of X, meaning those that cannot be weakly perturbed.

Theorem 4.7. *Let $I : X \to \mathbb{R}$ be a functional continuous with respect to the strong topology. Assume that*

$$\forall\, u \in X_0 \quad \exists\, u_k \in X_0 \text{ with } u_k \rightharpoonup u \text{ in } L^2,\ I(u_k) \to 0.$$

Then $S \subset \{u \in X : I(u) = 0\}$. In particular $\{u \in X : I(u) = 0\}$ is dense.

Proof. Let $u \in S$. By density there exists $u_k \in X_0$ such that $u_k \rightharpoonup u$. For each k there exists a sequence $u_{k,j} \in X_0$ by assumption such that $u_{k,j} \rightharpoonup u_k$ as $j \to \infty$ and $I(u_{k,j}) \to 0$. In particular by taking a diagonal sequence we obtain a sequence $\tilde{u}_k \in X_0$ such that $\tilde{u}_k \rightharpoonup u$ and $I(\tilde{u}_k) \to 0$. But $u \in S$, therefore $\tilde{u}_k \to u$ and $I(\tilde{u}_k) \to I(u)$. Hence $I(u) = 0$. □

In fact the approximation property (34) can also be weakened to a perturbation property:

Theorem 4.8. *Let $I : X \to \mathbb{R}_+$ be a functional continuous with respect to the strong topology. Assume that*

$$\forall\, u \in X_0 \text{ with } I(u) > 0 \quad \exists\, u_k \in X_0 \text{ such that}$$
$$u_k \rightharpoonup u \text{ in } L^2 \text{ and } \liminf_{k\to\infty} \|u_k\|_2^2 \geq \|u\|_2^2 + \alpha,$$

where $\alpha > 0$ depends only on $I(u) > 0$. Then $S \subset \{u \in X : I(u) = 0\}$. In particular $\{u \in X : I(u) = 0\}$ is dense.

Proof. Let $u \in S$, and assume that $I(u) > 0$. By density there exists $u_k \in X_0$ such that $u_k \rightharpoonup u$, and since $u \in S$, we have $u_k \to u$ strongly, and in particular $I(u_k) \to I(u)$. Then - by assumption - there exists $\alpha > 0$ (depending only on $I(u)$), such that for each k there exists a sequence $u_{k,j} \in X_0$ such that $u_{k,j} \rightharpoonup u_k$ as $j \to \infty$ and

$$\liminf_{j\to\infty} \|u_{k,j}\|_2^2 \geq \|u_k\|_2^2 + \alpha.$$

But then a suitable diagonal sequence $\tilde{u}_k := u_{k,j(k)} \in X_0$ satisfies $\tilde{u}_k \rightharpoonup u$ and

$$\liminf_{k\to\infty} \|\tilde{u}_k\|_2^2 \geq \|u\|_2^2 + \alpha,$$

contradicting the assumption that $u \in S$. □

5. Convex Integration

In this section we show how to apply the abstract ideas from Section 4 to produce (many) solutions to various problems. Before coming to the general statement in Section 5.3, we first look at differential inclusions for Lipschitz mappings as this is the situation that has been most extensively been looked at in the literature.

5.1. **Differential inclusions for Lipschitz mappings.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, i.e. an open bounded set with $|\partial\Omega| = 0$, and let $K \subset \mathbb{R}^{m\times n}$ be a compact set of matrices.

Let $\Gamma \subset \overline{\Omega}$ be a closed set with $|\Gamma| = 0$ and let $u_0 \in \mathrm{Lip}(\Gamma;\mathbb{R}^m)$. Consider the "Dirichlet-problem" for $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$:

$$\begin{cases} \nabla u(x) \in K & \text{a.e. } x \in \Omega \\ u(x) = u_0(x) & \text{for } x \in \Gamma. \end{cases} \tag{42}$$

Isometric maps for the flat metric $g = Id$ correspond to $K = O(n,m)$.

Assume that there exists an open set $U \subset \mathbb{R}^{m\times n}$ such that

$$(\mathcal{A}) \qquad \begin{aligned} &\forall\, A \in U \quad \exists\, u_k \in C_c^\infty(Q) \text{ such that} \\ &\qquad \text{(i)}\ \ A + \nabla u_k(x) \in U \quad \text{ for all } x \in Q, \\ &\qquad \text{(ii)}\ \int_Q \mathrm{dist}\,(A + \nabla u_k(x), K)\, dx \to 0 \quad \text{ as } k \to \infty, \end{aligned}$$

where $Q \subset \mathbb{R}^n$ is the open unit cube. Define X_0 and I as

$$\begin{aligned} X_0 &= \left\{ u \in C^\infty(\overline{\Omega}) : \nabla u(x) \in U \text{ for } x \in \overline{\Omega} \setminus \Gamma \text{ and } u|_\Gamma = u_0 \right\}, \\ I(u) &= \int_\Omega \mathrm{dist}\,(\nabla u(x), K)\, dx, \end{aligned}$$

and let X be the closure of X_0 in the uniform topology. It is not difficult to see that any set U with the property ($\mathcal{A}$) is a subset of the convex hull of K. Indeed, observe that any $A \in U$ and $u_k \in C_c^\infty(Q)$ defines a probability measure ν_k on $\mathbb{R}^{m\times n}$ as

$$\int_{\mathbb{R}^{m\times n}} f(\xi)\, d\nu_k(\xi) = \int_Q f(A + \nabla u_k(x))\, dx$$

with barycenter A and support $\mathrm{supp}\,\nu_k \subset U$. Then property (ii) amounts to

$$\int_{\mathbb{R}^{m\times n}} \mathrm{dist}\,(\xi, K)\, d\nu_k(\xi) \to 0 \quad \text{ as } k \to \infty.$$

Since K is compact we deduce that $\sup_k \int |\xi|\, d\nu_k(\xi) < \infty$, hence there exists a subsequence (not relabelled) such that $\nu_k \overset{*}{\rightharpoonup} \nu$ for some probability measure ν. But then ν has barycenter and $\int \mathrm{dist}\,(\xi, K)\, d\nu(\xi) = 0$ so that $\mathrm{supp}\,\nu \subset K$. This implies that A is contained in the convex hull of K.

In particular X is bounded in $W^{1,\infty}(\Omega)$ and therefore it is a compact metric space, where I is a Baire-1 functional. Moreover, an easy covering argument shows that

$$\forall\, u \in X_0 \quad \exists\, u_k \in X_0 \text{ with } u_k \to u \text{ in } X \text{ and } I(u_k) \to 0.$$

Therefore, as in Theorem 4.7 we deduce that $\{u \in X : I(u) = 0\}$ is residual in X.

In the literature condition ($\mathcal{A}$) is known as

U has the relaxation property with respect to K

see [DM97], as well as

U can be reduced to K

see [MS01]. In the example $K = O(n,m)$ we can take $U = \mathrm{int}\, K^{co} = \{A \in \mathbb{R}^{m\times n} : A^T A < I\}$, but in general U is forced to be considerably smaller.

An important point here is to understand the boundary condition $u|_\Gamma = u_0$. More precisely, we need to be able to check whether X_0 is nonempty: i.e. whether there exists a (smooth) extension of u_0 to $\overline{\Omega}$ such that $\nabla u_0(x) \in U$ for all $x \in \overline{\Omega} \setminus \Gamma$.

Recall from the proof of Theorem 3.5 that for the problem $\nabla u \in K := O(n,m)$ we would verify condition ($\mathcal{A}$) for $U = \operatorname{int} K^{co}$ by "adding" successively primitive metrics. This requires a decomposition of the metric error as in Lemma 3.3. In fact it suffices to be able to add just one primitive metric. More precisely, we can replace condition ($\mathcal{A}$) by

$$(\mathcal{P}) \qquad \begin{array}{c} \forall\, A \in U \text{ with } \operatorname{dist}(A,K) > \varepsilon \quad \exists\, u \in C_c^\infty(Q) \text{ such that} \\ \text{(i) } A + \nabla u(x) \in U \quad \text{ for all } x \in Q, \\ \text{(ii) } \displaystyle\int_Q |\nabla u(x)|^2\,dx > \delta, \end{array}$$

where $\delta = \delta_\varepsilon > 0$ only depends on $\varepsilon > 0$ but not on $A \in U$. This condition was introduced in [Kir03] as

gradients in U are stable only near K.

By using again a covering argument, we can deduce from ($\mathcal{P}$)

$$\begin{array}{c} \forall\, u \in X_0 \text{ with } I(u) \geq \alpha > 0 \quad \exists\, u_k \in X_0 \text{ such that} \\ u_k \to u \text{ in } X \text{ and } I(u_k) \leq I(u) - \beta, \end{array}$$

where $\beta = \beta_\alpha > 0$. As in Theorem 4.8 this implies once again that $\{u \in X : I(u) = 0\}$ is residual in X.

5.2. **Unit-length divergence-free fields.**

The isotropic case.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists $m \in L^\infty(\Omega)$ such that

$$\begin{aligned} \operatorname{div} \mathbf{1}_\Omega m &= 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ |m| &= 1 \text{ a.e. in } \Omega. \end{aligned}$$

Using the divergence theorem we have

$$\int_\Omega m \cdot \nabla\varphi = -\int_\Omega \varphi \operatorname{div} m + \int_{\partial\Omega} \varphi m \cdot \nu$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, so that the requirement $\operatorname{div} \mathbf{1}_\Omega m = 0$ is the weak formulation of

$$\operatorname{div} m = 0 \text{ in } \Omega,\ m \cdot \nu = 0 \text{ on } \partial\Omega.$$

Let

$$X_0 := \{m \in C^\infty(\Omega) : \operatorname{div} \mathbf{1}_\Omega m = 0,\ |m| < 1 \text{ in } \Omega\}.$$

We want to use Theorem 4.8, hence it suffices to prove:

Lemma 5.1. *For all $m \in X_0$ and all $\tilde{\Omega} \subset\subset \Omega$ there exists a sequence $m_k \in X_0$ such that*

$$\begin{aligned} m_k &\overset{*}{\rightharpoonup} m \text{ in } L^\infty(\Omega) \\ \liminf_{k\to\infty} \int_\Omega |m_k|^2 &\geq \int_\Omega |m|^2 + c\Big(\int_{\tilde{\Omega}} (1 - |m|^2)\Big)^2, \end{aligned}$$

where $c > 0$ is independent of m and $\tilde{\Omega}$.

Proof. Let $\xi, \eta \in \mathbb{R}^3$ with $|\xi| = |\eta| = 1$ and $u(x) = \eta \sin(x \cdot \xi)$. Then

$$\text{curl } u(x) = \nabla \times u(x) = (\xi \times \eta) \cos(x \cdot \xi).$$

Furthermore, for any $k \in \mathbb{N}$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$, if $u_k(x) = \frac{1}{k}\varphi(x)\eta \sin(kx \cdot \xi)$, then

$$\text{curl } u_k(x) = (\xi \times \eta)\varphi(x) \cos(kx \cdot \xi) + \frac{1}{k}(\nabla\varphi(x) \times \eta) \sin(kx \cdot \xi).$$

Let us now apply this formula with

$$\varphi(x) = \frac{1}{2}(1 - |m(x)|^2)\psi(x),$$

where $\psi \in C_c^\infty(\Omega)$ such that $0 \le \psi(x) \le 1$ in Ω and $\psi(x) = 1$ on $\tilde{\Omega}$. Also, let

$$m_k := m + \text{curl } u_k.$$

We claim that

(a) $m_k \in X_0$ for sufficiently large $k \in \mathbb{N}$,
(b) $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(\Omega)$ as $k \to \infty$,
(c) $\liminf_{k\to\infty} \int_\Omega |m_k|^2 \ge \int_\Omega |m|^2 + \frac{1}{8}\int_{\tilde{\Omega}}(1 - |m|^2)^2$.

To prove (a) note that $|m(x)| < 1$ for all $x \in \Omega$, hence

$$|m_k(x)| \le |m(x)| + \frac{1}{2}(1 - |m(x)|)(1 + |m(x)|) + C\frac{1}{k}$$

for some C depending on $\|m\|_{C^1}$ and $\|\psi\|_{C^1}$. Furthermore, as $\operatorname{supp}\psi \subset\subset \Omega$, there exists a $\delta > 0$ such that $|m(x)| \le 1 - \delta$ on $\operatorname{supp}\psi$. Hence, for $x \in \operatorname{supp}\psi$

$$\begin{aligned} |m_k(x)| &\le |m| + (1 - |m|)(1 - \delta/2) + C\frac{1}{k} \\ &\le 1 - \frac{\delta}{2}(1 - |m(x)|) + C\frac{1}{k} \le 1 - \frac{\delta^2}{2} + C\frac{1}{k} < 1 \end{aligned}$$

provided k is sufficiently large. Moreover, clearly $|m_k(x)| = |m(x)| < 1$ for $x \in \Omega \setminus \operatorname{supp}\psi$. This proves (a).

The claim (b) is obvious. Finally, claim (c) is precisely as in Example 1. The statement of the lemma now follows from (c) using Cauchy-Schwarz. □

The anisotropic case.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let also $K \subset \mathbb{R}^3$ be a compact set such that $0 \in \text{int } K^{co}$. Then there exists $m \in L^\infty(\Omega)$ such that

$$\begin{aligned} \operatorname{div} \mathbf{1}_\Omega m &= 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ m(x) &\in K \text{ a.e. in } \Omega. \end{aligned}$$

Remark 1. *The isotropic case corresponds to $K = S^2$. A typical anisotropic set could be $K = \{\pm e_1, \pm e_2, \pm e_3\}$, where e_1, e_2, e_3 are the coordinate unit vectors in $\mathbb{R}^3$. Observe that in this case the fact that $m(x) \in K$ almost everywhere is not compatible with the "classical" requirement that $m(x)$ should be tangential to the boundary of the domain $\partial\Omega$, and only the weak formulation $\operatorname{div} \mathbf{1}_\Omega m = 0$ makes sense.*

As before, let

$$X_0 := \{m \in C^\infty(\Omega) : \operatorname{div} \mathbf{1}_\Omega m = 0,\, m(x) \in \operatorname{int} K^{co} \text{ in } \Omega\}.$$

It is important to note that X_0 is not empty, precisely because of the requirement that $0 \in \operatorname{int} K^{co}$. Let

$$I(m) = \int_\Omega \operatorname{dist}^2(m, K).$$

Lemma 5.2. *For all $m \in X_0$ there exists a sequence $m_k \in X_0$ such that*

$$m_k \overset{*}{\rightharpoonup} m \text{ in } L^\infty(\Omega) \tag{43}$$

$$\liminf_{k\to\infty} \int_\Omega |m_k|^2 \geq \int_\Omega |m|^2 + \alpha, \tag{44}$$

where $\alpha > 0$ is only depending on $I(m) > 0$.

Our aim is to consider again perturbations of the form

$$m_k(x) = m(x) + \operatorname{curl}\left(\frac{1}{k}\eta(x)\sin\big(kx\cdot\xi(x)\big)\right),$$

where now the vectors $\eta, \xi \in \mathbb{R}^3$ are allowed to depend on the point $x \in \Omega$. Such a choice of perturbation guarantees (43). The key point is then to choose $\eta(x)$ and $\xi(x)$ appropriately. Note that - provided k is large - we have

$$m_k(x) \sim m(x) + \big[(\xi + (D\xi)^T x) \times \eta\big]\cos\big(kx\cdot\xi(x)\big).$$

The new term $(D\xi)^T x \times \eta$ can be dropped by localizing: e.g. take $\eta, \xi \in C_c^\infty$ such that ξ is constant on $\operatorname{supp}\eta$.

Next, note that for any vector $\overline{m} \in \mathbb{R}^3$ there exists $\eta, \xi \in \mathbb{R}^3$ such that $\overline{m} = \xi \times \eta$, so that we may assume that m_k has the form

$$m_k(x) \sim m(x) + \overline{m}\cos\big(kx\cdot\xi\big)$$

in a localized neighbourhood. There are now two constraints on $\overline{m}$: (i) not too big, so that $m_k(x) \in \operatorname{int}K^{co}$, but (ii) not too small, since

$$\liminf_{k\to\infty} \int_\Omega |m_k|^2 \sim \int_\Omega |m|^2 + \frac{1}{2}\int_\Omega |\overline{m}|^2,$$

and (44) should hold. The fact that (i) and (ii) can be simultaneously achieved by a good choice of $\overline{m}$ has nothing to do with divergence–free fields, it is a simple geometric fact about convex hulls:

Lemma 5.3 (Geometric lemma)**.** *Let $K \subset \mathbb{R}^d$ be compact. For any $\bar{z} \in$ int K^{co} there exists $\hat{z} \in \mathbb{R}^d$ such that*

$$\begin{aligned} \bar{z} + t\hat{z} &\in \textit{int } K^{co} \quad \textit{for all } t \in [-1, 1], \\ |\hat{z}| &\geq \frac{1}{2d}\operatorname{dist}(\bar{z}, K) \end{aligned}$$

Proof. Since $\bar{z} \in \operatorname{int} K^{co}$, by Carathéodory's theorem $\bar{z}$ is contained in the interior of a simplex spanned by elements of K, i.e.

$$\bar{z} = \sum_{i=1}^{d+1} \lambda_i z_i$$

with $0 < \lambda_i < 1$, $\sum_i \lambda_i = 1$ and $z_i \in K$. We may assume also that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d+1}$. Then it can be checked directly that

$$\bar{z} \pm \frac{1}{2}\lambda_j(z_1 - z_j) \in \text{int } K^{co} \quad \text{ for } j = 2, \ldots, d+1.$$

On the other hand $\bar{z} - z_1 = \sum_{i=2}^{d+1} \lambda_i(z_i - z_1)$, hence

$$\text{dist}\,(\bar{z}, K) \leq |\bar{z} - z_1| \leq d \max_{i=2,\ldots,d+1} \lambda_i |z_i - z_1|.$$

Choose $\hat{z} = \frac{1}{2}\lambda_j(z_1 - z_j)$ for j giving the maximum. Then

$$\frac{1}{2d}\,\text{dist}\,(\bar{z}, K) \leq |\hat{z}|,$$

concluding the proof.

□

Now we return to the divergence–free fields:

Proof of Lemma 5.2. Let $x_0 \in \Omega$, $r > 0$, and let $\psi \in C_c^\infty(B_r(x_0))$ be a cut–off function such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $B_{r/2}(x_0)$. Using the geometric lemma above, there exists $\eta, \xi \in \mathbb{R}^3$ such that if $u_k(x) = \frac{1}{k}\psi(x)\eta\sin(kx \cdot \xi)$, then

$$\begin{aligned} m(x_0) + \text{curl } u_k(x) &\in \text{int } K^{co} \quad \text{ for all } x \in \mathbb{R}^3 \text{ for sufficiently large } k \\ |\xi \times \eta| &\geq c\,\text{dist}\,\big(m(x_0), K\big). \end{aligned}$$

Moreover, since m is continuous, by choosing $r > 0$ smaller if necessary, we can ensure that

$$\begin{aligned} m(x) + \text{curl } u_k(x) &\in \text{int } K^{co} \quad \text{ for all } x \in \Omega \text{ for sufficiently large } k \\ \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |\text{curl } u_k|^2 &\geq c\,\text{dist}\,\big(m(x_0), K\big)^2. \end{aligned}$$

Now using the uniform continuity of m in Ω to find a finite family of pairwise disjoint balls $B_{r_j}(x_j) \subset \Omega$, vectors $\xi_j, \eta_j \in \mathbb{R}^3$ and cut–off functions $\psi_j \in C_c^\infty(B_{r_j}(x_j))$ such that if

$$u_k(x) = \sum_j \frac{1}{k}\eta_j \sin(kx \cdot \xi_j)\psi_j(x), \text{ and } m_k = m + \text{curl } u_k,$$

then $m_k \in X_0$ for sufficiently large k, and

$$\int_\Omega \text{dist}^{\,2}(m, K) \leq 2\sum_j |B_{r_j}(x_j)|\text{dist}^{\,2}(m(x_j), K) \leq c\int_\Omega |\text{curl } u_k|^2$$

In particular

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega |m_k|^2 &\geq \int_\Omega |m|^2 + \int_\Omega |\text{curl } u_k|^2 \\ &\geq \int_\Omega |m|^2 + c\int_\Omega \text{dist}^{\,2}(m, K), \end{aligned}$$

whereas on the other hand $m_k \overset{*}{\rightharpoonup} m$ in L^∞.

□

5.3. **The Tartar framework.** The methods presented so far all rely on the same basic principle: solutions are constructed from a suitable "subsolution" by successively adding highly oscillatory perturbations. The subsolutions can be seen as a relaxation of the original problem. Therefore the construction amounts to an "undoing" the relaxation. It is thus natural to look for a generalization of this method in the framework of compensated compactness, originally introduced by L. Tartar and R. DiPerna [Tar79, DiP85] to study possible compensation effects in the relaxation. Indeed, a rule of thumb is that compensated compactness applies to situations where the relaxation is *small* (e.g. there exists no non-empty open set U with properties ($\mathcal{A}$) or ($\mathcal{P}$) for the problem (42)), whereas a *large* relaxation leads to a residual set of solutions. The presentation in this section is from [Szé12].

We consider general systems in a bounded domain $\mathscr{D} \subset \mathbb{R}^d$ of the form

$$\sum_{i=1}^{d} A_i \partial_i z = 0 \qquad \text{in } \mathscr{D} \tag{45}$$

$$z(y) \in K \qquad \text{a.e. } y \in \mathscr{D} \tag{46}$$

where

$$z : \mathscr{D} \subset \mathbb{R}^d \to \mathbb{R}^N$$

is the unknown state variable, A_i are constant $m \times N$ matrices, and $K \subset \mathbb{R}^N$ is a given compact set.

The example of divergence-free fields discussed in Section 5.2 obviously fits in this framework. Also, the first order differential inclusions from Section 5.1 can be cast in the form (45)-(46) by observing that locally being a gradient is equivalent to being curl-free. Then, setting $d = n$, $y = x$, $\mathscr{D} = \Omega$, and $\mathbb{R}^N \cong \mathbb{R}^{m \times n}$ the differential inclusion (42) can be written as

$$\begin{aligned} \operatorname{curl} z = 0 &\quad \text{in } \mathscr{D} \\ z(x) \in K &\quad \text{a.e. } x \in \mathscr{D}, \end{aligned} \tag{47}$$

where $z : \Omega \to \mathbb{R}^{m \times n}$.

We make the following assumptions.

(H1) The Wave Cone: There exists a closed cone $\Lambda \subset \mathbb{R}^N$ and a constant $C > 0$ such that for all $\hat{z} \in \Lambda$ there exists a sequence $z_k \in C_c^\infty(B_1(0); \mathbb{R}^N)$ such that

- $\sum_{i=1}^d A_i \partial_i z_k = 0$ in $\mathscr{D}$;
- $\operatorname{dist}(z_k, [-\hat{z}, \hat{z}]) \to 0$ uniformly;
- $z_k \rightharpoonup 0$ weakly in L^2;
- $\int |z_k|^2 dy > C|\hat{z}|^2$.

(H2) The Λ-convex hull: There exists a bounded open set $U \subset \mathbb{R}^N$ with $U \cap K = \emptyset$, and such that for all $z \in U$ with $\operatorname{dist}(z, K) \geq \alpha > 0$ there exists $\hat{z} \in \Lambda \cap S^{N-1}$ such that

$$z + t\hat{z} \in U \text{ for all } |t| < \beta, \tag{48}$$

where $\beta = \beta(\alpha) > 0$.

(H3) Subsolutions: X_0 is a nonempty bounded subset of $L^2(\mathscr{D})$ consisting of functions which are "perturbable" in an open subdomain $\mathscr{U} \subset \mathscr{D}$. This means that any $z \in X_0$ is continuous on $\mathscr{U}$ with

$$z(y) \in U \quad \text{for } y \in \mathscr{U}\,, \tag{49}$$

and moreover, if $z \in X_0$ and $w \in C_c(\mathscr{U})$ such that w solves (45) and $(z+w)(y) \in U$ for all $y \in \mathscr{U}$, then $z + w \in X_0$.

Finally, let X be the closure of X_0 with respect to the weak L^2 topology. Since X_0 is bounded, the topology of weak L^2 convergence is metrizable on X, making it into a complete metric space.

Theorem 5.4. *Assuming (H1)-(H3), the set*

$$\{z \in X : z(y) \in K \ \text{a.e. } y \in \mathscr{U}\}$$

is residual in X.

The proof is a direct application of Theorem 4.8 with $I(z) = \int_{\mathscr{U}} \Psi(z(y))\,dy$ below together with the following statement.

Lemma 5.5. *There exists a continuous function*

$$\Psi : \overline{U} \to [0, \infty) \qquad \text{with } \{\Psi = 0\} \subset K \tag{50}$$

with the following property: For any $z \in X_0$ there exists a sequence $z_k \in X_0$ with $z_k \rightharpoonup z$ in L^2 and such that

$$\int_{\mathscr{U}} |z - z_k|^2 dy \geq \int_{\mathscr{U}} \Psi\big(z(y)\big)\,dy.$$

Proof. To start with, note that the hypotheses (H1)-(H2) together lead to the following statement: There exists a continuous function Ψ with (50) such that for any $\bar z \in U$ there exists $\tilde U \subset\subset U$ and a sequence $z_k \in C_c^\infty(B_1(0); \mathbb{R}^N)$ such that

(a) $\sum_{i=1}^d A_i \partial_i z_k = 0$ in $\mathscr{D}$;
(b) $\bar z + z_k(y) \in \tilde U$ for all y;
(c) $z_k \rightharpoonup 0$ weakly in L^2;
(d) $\int |z_k|^2 dy > 2\Psi(\bar z)$.

Fix $y_0 \in \mathscr{U}$, $r_0 > 0$ and let $\bar z = z(y_0)$. Applying the above to $\bar z$ together with the translation and rescaling $y \mapsto r_0^{-1}(y - y_0)$, we obtain a sequence $z_k \in C_c^\infty(B_{r_0}(y_0); \mathbb{R}^N)$ such that (a)-(c) holds and (d) is replaced by

(d') $\int_{B_{r_0}(y_0)} |z_k|^2 dy > 2|B_{r_0}(y_0)|\Psi\big(z(y_0)\big)$.

Moreover, using the fact that z is continuous, we may choose $r_0 = r_0(y_0) > 0$ sufficiently small so that (b) can be replaced by

(b') $z(y) + z_k(y) \in U$ for all y.

Using a domain exhaustion argument, we then find disjoint balls $B_i := B_{r_i}(y_i) \subset\subset \mathscr{U}$ for $i = 1 \dots I$ and associated sequences z_k^i such that (a),(b'),(c),(d') hold for each i, k and

$$\int_{\mathscr{U}} \Psi\big(z(y)\big)\,dy \leq 2\sum_{i=1}^{I} |B_i|\Psi\big(z(y_i)\big).$$

Set

$$z_k := z + \sum_{i=1}^{I} z_k^i.$$

It is easy to see that $z_k(y) \in U$ for all $y \in \mathscr{U}$, hence by (H3), $z_k \in X_0$. Furthermore $z_k \rightharpoonup z$ in L^2 and

$$\begin{aligned}\int_{\mathscr{U}} |z_k - z|^2\,dy &= \sum_{i=1}^{I} \int_{B_i} |z_k^i|^2\,dy \\ &\geq 2\sum_{i=1}^{I} |B_i|\Psi\big(z(y_i)\big) \\ &\geq \int_{\mathscr{U}} \Psi\big(z(y)\big)\,dy.\end{aligned}$$

This concludes the proof. □

For first order partial differential inclusions written in the form (47) Λ is given by the rank-one cone

$$\Lambda = \{z \in \mathbb{R}^{m\times n} : \text{rank } z \leq 1\}$$

and hypothesis (H2) amounts precisely to the perturbation condition $(\mathcal{P})$. More generally a candidate for the cone in (H1) is the wave cone, given by one-dimensional solutions of (45) as

$$\Lambda = \left\{ z \in \mathbb{R}^N : \left(\sum_{i=1}^{d} \xi_i A_i\right) z = 0 \text{ for some } \xi \in S^{d-1} \right\} \tag{51}$$

However, in this generality the existence of compactly supported functions as in (H1) would require the constant-rank condition on the differential operator in (45) (see for instance [FM99]), which fails to hold in certain cases, e.g. for the incompressible Euler equations. See also [Shv11] for an example where restricting the wave cone is useful.

Concerning (H2) observe that:

Lemma 5.6. *If bounded sets $K, U \subset \mathbb{R}^N$ satisfy (H2), then $U \subset K^{co}$.*

Proof. Assuming that U is not contained in the convex hull of K, there exists $z_0 \in \overline{U}$ and $w \in \mathbb{R}^N$ such that

$$\sup_{z\in K} w\cdot z < w \cdot z_0.$$

Since U is bounded, without loss of generality we may assume that $w \cdot z_0 = \sup_{z\in U} w \cdot z =: \gamma_0$. Let

$$K_1 := \{z \in \overline{U} : w \cdot z = \gamma_0\} \neq \emptyset.$$

Since K and K_1 are compact, $\inf_{z\in K} \text{dist}\,(z, K_1) \geq \alpha$ for some $\alpha > 0$. Therefore, by (H2) there exists $\beta > 0$ such that for any $z \in K_1$ there exists $\hat{z} \in \Lambda \cap S^{N-1}$ with $z + t\hat{z} \in \overline{U}$ for all $|t| \leq \beta$. Since $w \cdot z \leq \gamma_0$ for all $z \in \overline{U}$, it follows that $w \cdot \hat{z} = 0$, so that

$$z + t\hat{z} \in K_1 \text{ for all } |t| \leq \beta.$$

This implies that K_1 has no (convex) extreme points, a contradiction with the classical theorem of Minkowski implying that the compact set K equals the convex hull of its extreme points. $\square$

Consequently a candidate for U could be the interior of the convex hull of K. However, in many situations this set is too large and does not satisfy (H2), because the wave cone Λ is too small. In the context of first order differential inclusions, where Λ is the rank-one cone, there are many techniques available for studying the possible sets U - these are in general related to the *rank-one convex hull* of K. We refer the reader to [KMŠ03, Kir03].

6. Euler Subsolutions

Here we show how to apply the ideas from Section 5 to the incompressible Euler equations, in particular to the results from Section 2.

6.1. **The Reynolds stress and subsolutions.** The proof of Theorems 2.1-2.2 as well as Theorems 2.4-2.5 is based on the notion of *subsolution*. Subsolutions should be thought as the analogue of short maps for the embedding problem in Section 3.

In order to motivate the definition, let us recall the concept of Reynolds stress. It is generally accepted that the appearance of high-frequency oscillations in the velocity field is the main reason responsible for turbulent phenomena in incompressible flows. One related major problem is therefore to understand the dynamics of the coarse-grained, in other words macroscopically averaged, velocity field. If $\overline{v}$ denotes the macroscopically averaged velocity field, then it satisfies

$$\begin{array}{l}\partial_t \overline{v} + \operatorname{div}(\overline{v}\otimes\overline{v} + R) + \nabla\overline{p} = 0\\ \operatorname{div}\overline{v} = 0,\end{array} \tag{52}$$

where

$$R = \overline{v\otimes v} - \overline{v}\otimes\overline{v}\,.$$

The latter quantity is called Reynolds stress and arises because the averaging does not commute with the nonlinearity $v\otimes v$. On this formal level the precise definition of averaging plays no role, be it long-time averages, ensemble-averages or local space-time averages. The latter can be interpreted as taking weak limits. Indeed, weak limits of Leray solutions of the Navier-Stokes equations with vanishing viscosity have been proposed in the literature as a deterministic approach to turbulence (see [Lax91], [Cho94], [BGK00], [BT07]).

A slightly more general version of this type of averaging follows the framework introduced by Tartar [Tar79, Tar83] and DiPerna [DiP85] in the context of conservation laws. We start by separating the linear equations from the nonlinear constitutive relations. Accordingly, we write (52) as

$$\begin{aligned}\partial_t\overline{v} + \operatorname{div}\overline{u} + \nabla\overline{q} &= 0\\ \operatorname{div}\overline{v} &= 0,\end{aligned}$$

where $\overline{u}$ is the traceless part of $\overline{v}\otimes\overline{v}+R$. Since one can write

$$R = \overline{(v-\overline{v})\otimes(v-\overline{v})},$$

it is clear that $R\geq 0$, i.e. R is a symmetric positive semidefinite matrix. In terms of the coarse-grained variables $(\overline{v},\overline{u})$ this inequality can be written as

$$\overline{v}\otimes\overline{v} - \overline{u} \leq \tfrac{2}{n}\overline{e}\,I,$$

where I is the $n \times n$ identity matrix and

$$\overline{e} = \overline{\tfrac{1}{2}|v|^2}$$

is the macroscopic kinetic energy density. Motivated by these calculations, we define subsolutions as follows. Let us introduce the notation $\mathcal{S}_0^{n\times n}$ for the vector space of symmetric traceless $n \times n$ matrices.

Definition 6.1 (Subsolutions). *Let $\overline{e} \in L^1(\mathbb{T}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple*

$$(v, u, q) : \mathbb{T}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$$

such that $v \in L^2$, $u \in L^1$, q is a distribution,

$$\left\{ \begin{array}{l} \partial_t v + \operatorname{div} u + \nabla q = 0 \\ \operatorname{div} v = 0, \end{array} \right. \qquad \textit{in the sense of distributions,} \tag{53}$$

and moreover

$$v \otimes v - u \leq \tfrac{2}{n}\overline{e}\, I \qquad \textit{a.e.} \tag{54}$$

Observe that subsolutions automatically satisfy $\frac{1}{2}|v|^2 \leq \overline{e}$ a.e. (this follows from taking the trace in (54)). If in addition we have $\frac{1}{2}|v|^2 = \overline{e}$ a.e., then the v component of the subsolution is in fact a weak solution of the Euler equations. A convenient way to express the inequality (54) is obtained by introducing the *generalized energy density*

$$e(v, u) = \frac{n}{2}|v \otimes v - u|_\infty,$$

where $|\cdot|_\infty$ is the operator norm of the matrix(= the largest eigenvalue for symmetric matrices). The inequality (54) can then be equaivalently written as

$$e(v, u) \leq \overline{e} \text{ a.e.} \tag{55}$$

As mentioned above, in passing to weak limits (or when considering any other averaging process), the high-frequency oscillations in the velocity are responsible for the appearance of a non-trivial Reynolds stress. Equivalently stated, this phenomenon is responsible for the inequality sign in (54). The key point of convex integration is that a *strict inequality* instead of (54) gives enough room so that the high-frequency oscillations can be "added back" on top of the subsolution – of course in a highly non-unique way. This is the content of the following theorem (essentially Proposition 2 from [DLS10]).

Theorem 6.2 (Subsolution criterion)**.** *Let $\overline{e} \in L^\infty(\mathbb{T}^n \times (0,T))$ and $(\overline{v}, \overline{u}, \overline{q})$ be a subsolution. Furthermore, let $\mathscr{U} \subset \mathbb{T}^n \times (0,T)$ a subdomain such that $(\overline{v}, \overline{u}, \overline{q})$ and $\overline{e}$ are continuous on $\mathscr{U}$ and*

$$\begin{aligned} e(\overline{v}, \overline{u}) &< \overline{e} && \textit{on } \mathscr{U} \\ e(\overline{v}, \overline{u}) &= \overline{e} && \textit{a.e. } \mathbb{T}^n \times (0,T) \setminus \mathscr{U} \end{aligned} \tag{56}$$

Then there exist infinitely many weak solutions $v \in L^\infty(0,T; L^2(\mathbb{T}^n))$ of the Euler equations such that

$$\begin{aligned} v &= \overline{v} && \textit{almost everywhere outside } \mathscr{U} \\ \tfrac{1}{2}|v|^2 &= \overline{e} && \textit{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \end{aligned}$$

Condition (56) amounts to the requirement that in the open subset $\mathscr{U} \subset \mathbb{T}^n \times (0,T)$ where $(\overline{v}, \overline{u}, \overline{q})$ is not a solution, it should actually be a strict subsolution. Let us further remark that:

- If in addition

$$\overline{v}(\cdot, t) \rightharpoonup v_0(\cdot) \text{ in } L^2(\mathbb{T}^n) \text{ as } t \to 0, \tag{57}$$

  then all the v's so constructed solve the Cauchy problem (1).
- Furthermore, the construction can be done in such a way that the pressure for such solutions satisfies

$$p = \overline{q} - \tfrac{2}{n}\overline{e} \quad \text{ for a.e. } (x,t). \tag{58}$$

In this formulation the Euler equations have been cast in the general framework (45)-(46) of Tartar, with

$$\begin{aligned}
d &= n+1,\\
y &= (x,t),\\
\mathscr{D} &= \mathbb{T}^n \times (0,T),\\
z &= (v,u,q),\\
\mathbb{R}^N &\cong \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R},\\
K &= \big\{(v,u,q) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R} :\, e(v,u) = \overline{e}\big\},
\end{aligned}$$

and (45) given by (53). Observe that in this case $K = K_{\overline{e}}$ depends on $(x,t) \in \mathscr{D}$ (unless $\overline{e}$ is constant). Nevertheless, the theory developed in Section 5.3 still applies with only minor modifications.

Condition (H1).

We start by calculating the wave cone as defined in (51) for the Euler system (53). This amounts to finding plane-wave solutions of the form

$$(v,u,q)(x,t) = (\hat{v}, \hat{u}, \hat{q}) \sin(x \cdot \xi + ct),$$

where $(\hat{v}, \hat{u}, \hat{q}) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$. We are lead to

$$c\hat{v} + \hat{u}\xi + \hat{q}\xi = 0, \quad \xi \cdot \hat{v} = 0. \tag{59}$$

To solve this, introduce the projection $P_{\hat{v}} = I - \frac{\hat{v}\otimes\hat{v}}{|\hat{v}|^2}$ onto the subspace orthogonal to $\hat{v}$. Applying this projection, the equations can be written equivalently as

$$P_{\hat{v}}\, \hat{u}\, P_{\hat{v}}\, \xi = -\hat{q}\xi.$$

In other words, given any nonzero $\hat{v}$ and any $\hat{u}$, the triple $(\hat{v}, \hat{u}, \hat{q})$ is an element of the wave cone if (and only if) $-\hat{q}$ is an eigenvalue of the symmetric matrix $P_{\hat{v}}\hat{u}P_{\hat{v}}$.

Condition (H1) amounts to the following statement:

Lemma 6.3. *For any nonzero $\hat{v} \in \mathbb{R}^n$ and any $\hat{u} \in \mathcal{S}_0^{n\times n}$ there exists $\hat{q} \in \mathbb{R}$ and a sequence $z_k = (v_k, u_k, q_k) \in C_c^\infty(B_1(0))$ such that, setting $\hat{z} = (\hat{v}, \hat{u}, \hat{q})$ we have*

(a) (v_k, u_k, q_k) *solves*(53),
(b) $\operatorname{dist}(z_k, [-\hat{z}, \hat{z}]) \to 0$ *uniformly,*
(c) $z_k \rightharpoonup 0$ *weakly in* L^2,
(d) $\int |z_k|^2 dy > C|\hat{z}|^2$.

In order to prove Lemma 6.3, note that, by the very definition of the wave cone, the sequence

$$z_k(y) = \hat{z}\sin(ky\cdot\eta), \tag{60}$$

where $\eta = (\xi, c)$ is as in (59), satisfies (a)-(d), *but* z_k is not compactly supported. There are two ways to remedy this problem:

(1) If we multiply by a cut-off function $\varphi \in C_c^\infty(B_1(0))$, the function does not satisfy (53) any more. We could add a corrector $\zeta_k \in C_c^\infty(B_1(0);\mathcal{M})$, i.e. set

$$z_k(y) = \hat{z}\varphi(y)\sin(ky\cdot\xi) + \zeta_k(y), \tag{61}$$

such that

$$\operatorname{div}\zeta_k = f_k \tag{62}$$

where

$$f_k(y) = -\operatorname{div} z_k(y) = -\hat{z}\nabla\varphi(y)\,\sin(ky\cdot\xi).$$

Since $f_k \to 0$ in $W^{-1,\infty}$ it is natural to expect the estimate

$$\|\zeta_k\|_{C^0(B_1(0))} = o(1) \text{ as } k \to \infty,$$

leading to condition (b), provided we can solve (62) in a reasonable way. This argument becomes important in producing continuous weak solutions, see [DLS13, DLS12a].

(2) An alternative is to find a *potential*, analogously to the curl operator for divergence-free fields in Section 5.2. In fact it suffices to show that to any $\overline{M} \in \Lambda$ there exists a matrix–valued, constant coefficient, homogeneous linear differential operator

$$A(\partial) : C^\infty(\mathbb{R}^{n+1}) \to C^\infty\big(\mathbb{R}^{n+1}; \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}\big)$$

and a space-time vector $\eta \in \mathbb{R}^{n+1}$ such that for any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$

$$(v, u, q) = A(\partial)\phi \text{ satisfies (53)},$$

and if $\phi(y) = \psi(y\cdot\eta)$, then

$$A(\partial)\phi(y) = \hat{z}\,\frac{d^l\psi}{ds^l}(y\cdot\eta), \tag{63}$$

where $l \in \mathbb{N}$ is the order of A. With such a potential, we can set

$$z_k(y) = \frac{1}{k^l}A(\partial)\big[\psi(ky\cdot\eta)\varphi(y)\big].$$

Observe that by (63) and the product rule we can write

$$z_k(y) = \hat{z}\,\frac{d^l\psi}{ds^l}(y\cdot\eta)\varphi(y) + O\left(\frac{1}{k}\right), \tag{64}$$

where the $O(\frac{1}{k})$ term is small in the C^0 norm. Compare this to (61).

Such potentials can be constructed by observing that the matrix-valued polynomial $A = A(\xi)$ needs to satisfy

$$A^T = A, \quad A\xi = 0, \quad Ae_{n+1}\cdot e_{n+1} = 0.$$

Such matrices can be obtained from the formula

$$A(\xi) = \tfrac{1}{2}\left(R\xi \otimes Q\xi + Q\xi \otimes R\xi\right),$$

where R and Q are anti-symmetric $(n+1)\times(n+1)$ matrices such that $Re_{n+1}=0$. The condition (63) can be obtained by observing that for such A and $\phi(y)=\psi(y\cdot\eta)$ we have

$$A(\partial)\phi(y)=A(\eta)\,\frac{d^2\psi}{ds^2}(y\cdot\eta).$$

For further details, and for how to obtain *pressureless oscillations* which are needed for (58), see Proposition 4.8 in [DLS10]. An alternative way to obtain potentials can be found in [DLS09].

Condition (H2).

In Lemma 6.3 we saw that for any $(\hat v,\hat u)$ there exists a $\hat q$ so that the triple $(\hat v,\hat u,\hat q)$ is contained in the Λ-cone. On the other hand observe that the nonlinear constraint and the inequality defining subsolutions in (54) and (56) only involves the state variables (v,u) and not q. Consequently, in order to verify (H2) with K and U given by the equality and inequality in (56), respectively, it suffices to show that U is the (interior of the) convex hull of K. This is the content of the next lemma.

Lemma 6.4. *Let $\bar e>0$ and define*

$$K=\left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}_0^{n\times n}\,:\,e(v,u)=\bar e\right\},$$
$$U=\left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}_0^{n\times n}\,:\,e(v,u)<\bar e\right\}.$$

Then $K^{co}=\overline{U}$.

Proof. First of all observe that

$$(v,u)\mapsto e(v,u)\text{ is convex.}$$

Indeed, this follows from writing

$$e(v,u)=\frac{n}{2}\max_{\xi\in S^{n-1}}\left\langle\xi,(v\otimes v-u)\xi\right\rangle=\frac{n}{2}\max_{\xi\in S^{n-1}}\left[|\langle\xi,v\rangle|^2-\langle\xi,u\xi\rangle\right]$$

and noting that for every $\xi\in S^{n-1}$ the map $(v,u)\mapsto|\langle\xi,v\rangle|^2-\langle\xi,u\xi\rangle$ is convex. Consequently, $\overline{U}\supset K^{co}$.

To show equality we need to show that any extreme point (v_0,u_0) of $\overline{U}$ is contained in K. By a suitable rotation of the coordinate axes we may assume that $v_0\otimes v_0-u_0$ is diagonal, with diagonal entries $1\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$. If $\lambda_n=1$, then $v_0\otimes v_0-u_0=I$ so that $(v_0,u_0)\in K$.

Next, assume that $\lambda_n<1$. Let $e_1,\dots,e_n$ be the coordinate vector, write $v_0=\sum_{i=1}^n v^ie_i$ in coordinates and let

$$\overline{v}=e_n,\quad\overline{u}=\sum_{i=1}^{n-1}v^i(e_i\otimes e_n+e_n\otimes e_i).$$

Then $\overline{u}\in\mathcal{S}_0^{n\times n}$ and

$$(v_0+t\overline{v})\otimes(v_0+t\overline{v})-(u_0+t\overline{u})=v_0\otimes v_0-u_0+(2tv^n+t^2)e_n\otimes e_n.$$

Since $\lambda_n<1$, it follows that $(v_0,u_0)+t(\overline{v},\overline{u})\in\overline{U}$ for all sufficiently small $|t|$ and hence (v_0,u_0) cannot be an extreme point of $\overline{U}$. This concludes the proof. $\square$

Condition (H3) and the proof of Theorem 6.2.

Let $\bar{e}$, $(\bar{v}, \bar{u}, \bar{q})$ and $\mathscr{D}$ be as in the statement of Theorem 6.2. We define the approximating space of subsolutions X_0 as follows:

$$\begin{aligned} X_0 := \Big\{ (v,u,q) \text{ subsolution} : &\text{(i) } (v,u,q) = (\bar{v}, \bar{u}, \bar{q}) \text{ on } \mathscr{D} \setminus \mathscr{U}\,, \\ &\text{(ii) } (v,u,q) \text{ continuous on } \mathscr{U}\,, \\ &\text{(iii) } e(v,u) < \bar{e} \text{ in } \mathscr{U} \Big\}, \end{aligned}$$

where $(\bar{v}, \bar{u}, \bar{q})$ is the subsolution given in Theorem 6.2. Observe that the assumptions in the theorem guarantee that X_0 is nonempty.

Lemma 6.5. *X_0 is bounded in $L^2(\mathscr{D})$*

Proof. Observe first of all that (54) implies (by taking the trace)

$$\frac{1}{2}|v|^2 \le \bar{e} \quad \text{ for a.e. } (x,t) \in \mathscr{D}.$$

Hence $\|v\|^2_{L^\infty(\mathscr{D})} \le 2\|\bar{e}\|_{L^\infty(\mathscr{D})}$. Moreover, since u is symmetric and traceless, the operator matrix norm can be estimated as

$$|u|_\infty \le (n-1)|\lambda_{min}(u)|,$$

where $\lambda_{min}(u)$ is the smallest eigenvalue. In turn, for any $\xi \in S^{n-1}$ we have

$$-u\xi \cdot \xi = (v \otimes v - u)\xi \cdot \xi - |v\xi|^2 \le \frac{2\bar{e}}{n},$$

so that $\|u\|_{L^\infty(\mathscr{D})} \le 2\|\bar{e}\|_{L^\infty(\mathscr{D})}$. Finally, from (53) we get

$$-\Delta q = \operatorname{div}\operatorname{div} u \quad \text{ on } \mathbb{T}^n$$

so that in particular $\|q\|_{L^2(\mathscr{D})} \le \|u\|_{L^2(\mathscr{D})}$. It follows that X_0 is bounded in $L^2(\mathscr{D})$, by a bound only depending on $\|\bar{e}\|_{L^\infty(\mathscr{D})}$. □

Let X be the closure of X_0 in the weak $L^2(\mathscr{D})$ topology. Then X is a compact metric space as in Section 5.3 and X_0 is perturbable in the sense of hypothesis (H3). The statement of the theorem is then a direct consequence of Theorem 5.4 together with the considerations in (H1) and (H2) above with

$$\begin{aligned} K &= \left\{ (v,u) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} :\, e(v,u) = \bar{e} \right\}, \\ U &= \left\{ (v,u) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} :\, e(v,u) < \bar{e} \right\}. \end{aligned}$$

Note that $K = K_{\bar{e}}$ and $U = U_{\bar{e}}$ depend on (x,t) through the dependence on $\bar{e} = \bar{e}(x,t)$. However, it is not difficult to see that, since $\bar{e}$ is assumed to be continuous on $\mathscr{U}$, it suffices to check (H2) for $U_{\bar{e}}$ and $K_{\bar{e}}$ for each *fixed* $\bar{e} > 0$ and then apply an additional covering argument.

6.2. **Construction of subsolutions.** Using Theorem 6.2, the results of Section 2 can be reduced to showing the existence of a suitable subsolution.

Compact support in time; Theorem 2.1.

Fix $0 < T_1 < T_2 < T$. Choose $(\bar{v}, \bar{u}, \bar{q}) \equiv 0$ and

$$\bar{e} = \begin{cases} 1 & t \in (T_1, T_2), \\ 0 & \text{otherwise}. \end{cases}$$

Then the conditions of Theorem 6.2 are satisfied with $\mathcal{U} = \mathbb{T}^n \times (T_1, T_2)$. The solutions v obtained in this way have compact support in time. This in particular proves Theorem 2.1.

Fixed initial data I; Theorem 2.2.

In the example above the initial datum is trivially seen to be zero. In the same way any other fixed initial data can be prescribed by only using perturbations which are compactly supported in time.

For Theorem 2.2, i.e. without requiring admissibility, this can be done by *first* constructing a smooth solution $(\bar{v}, \bar{u}, \bar{q})$ to the underdetermined Cauchy problem

$$\begin{aligned} \partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} &= 0, \\ \operatorname{div} \bar{v} &= 0, \\ \bar{v}|_{t=0} &= v_0, \end{aligned} \tag{65}$$

and *then* choosing $\bar{e}$ so that $(\bar{v}, \bar{u}, \bar{q})$ becomes a strict subsolution with respect to $\bar{e}$. Hence we want to choose the function $\bar{e}$ so that $\bar{e} > e(\bar{v}, \bar{u})$. Since the solutions should remain in the energy space, we also require $\bar{e} \in L^\infty(0, T; L^2(\mathbb{T}^n))$, which in turn requires

$$\bar{v} \in L^\infty(0, T; L^2(\mathbb{T}^n)), \quad \bar{u} \in L^\infty(0, T; L^1(\mathbb{T}^n)).$$

Such a solution can be found for instance by setting $\bar{v}$ to be the solution of

$$\begin{aligned} \partial_t \bar{v} + (-\Delta)^{1/2} \bar{v} &= 0, \\ \bar{v}|_{t=0} &= v_0, \end{aligned}$$

and then setting $\bar{u}_{ij} = -\mathcal{R}_j \bar{v}_i - \mathcal{R}_i \bar{v}_j$, where $\mathcal{R}$ is the (vector-valued) Riesz transform with Fourier symbol $i\frac{\xi}{|\xi|}$. For further details we refer to [Wie11].

Fixed initial data II; Theorem 2.4.

If we require admissibility, we need to find (smooth) solutions of the Cauchy problem (65) satisfying the additional requirement

$$\int_{\mathbb{T}^n} e\left(\bar{v}(x,t), \bar{u}(x,t)\right) dx < \frac{1}{2} \int_{\mathbb{T}^n} |v_0(x)|^2 \, dx \qquad \text{for } t > 0$$

so that we can define $\bar{e}$ such that

- $(\bar{v}, \bar{u}, \bar{q})$ is a strict subsolution for $t > 0$;
- $\int_{\mathbb{T}^n} \bar{e}(x, 0) \, dx = \frac{1}{2} \int_{\mathbb{T}^n} |v_0(x)|^2 \, dx$.

An example, where this can be done in such a way that the solutions obtained will be admissible, is given by the shear flow initial data in 2D, Theorem 2.4. Thus, set v_0 to be the periodic extension of

$$v_0(x_1, x_2) = \begin{cases} e_1 & \text{if } x_2 \in (0, \frac{1}{2}), \\ -e_1 & \text{if } x_2 \in (-\frac{1}{2}, 0). \end{cases} \tag{66}$$

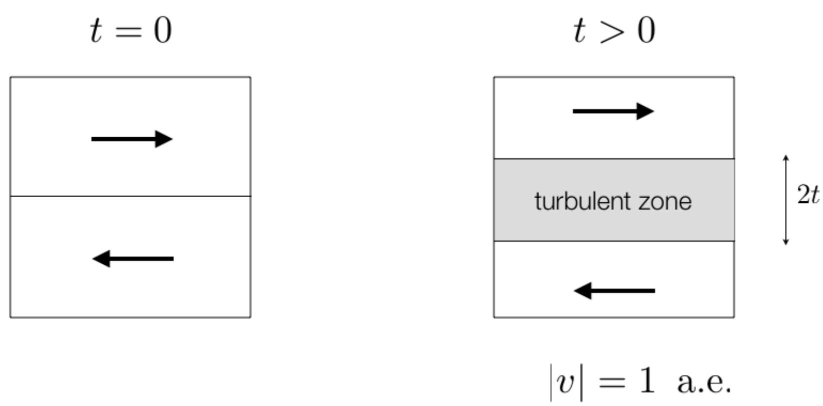

FIGURE 3. Shear flow initial datum with a "turbulent zone" appearing for $t > 0$.

We set

$$\bar{v} := (\alpha, 0), \quad \bar{u} := \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \quad \bar{q} := \beta$$

where $\alpha = \alpha(x_2, t)$, $\beta = \beta(x_2, t)$ and $\gamma = \gamma(x_2, t)$ are functions still to be fixed. With these choices the system (65) reduces to

$$\partial_t \alpha + \partial_{x_2} \gamma = 0. \tag{67}$$

whereas

$$\bar{e}I - \bar{v} \otimes \bar{v} + \bar{u} = \begin{pmatrix} \bar{e} - \alpha^2 + \beta & \gamma \\ \gamma & \bar{e} - \beta \end{pmatrix}.$$

Next, for some $\lambda \in (0, 1)$ we set $\beta := \frac{1}{2}\alpha^2$ and $\gamma := -\frac{\lambda}{2}(1 - \alpha^2)$. With this choice (67) becomes the inviscid Burgers equation $\partial_t \alpha + \frac{\lambda}{2} \partial_{x_2} \alpha^2 = 0$, with (1-periodic) initial data given by $\alpha(x_2, 0) = s(x_2)$. Set α to be the (unique) viscosity solution, given by a rarefaction wave with speed λ at $x_2 = 0$ and constant shocks at $x_2 = \pm\frac{1}{2}$ upto time

$$T = \tfrac{1}{2\lambda},$$

i.e. the 1-periodic extension of $x_2 \mapsto \alpha(x_2, t)$ given by

$$\alpha(x_2, t) := \begin{cases} -1 & -\frac{1}{2} < x_2 < -\lambda t, \\ \frac{x_2}{\lambda t} & -\lambda t < x_2 < \lambda t, \\ 1 & \lambda t < x_2 < \frac{1}{2}, \end{cases} \qquad \text{for } t < T. \tag{68}$$

In particular, we have $|\alpha| \leq 1$ for all (x, t). Set

$$\mathcal{U} := \{(x, t) : |x_2| < \lambda t\}$$

and

$$\bar{e}(x,t) := \frac{1}{2} - \varepsilon \frac{1-\lambda}{2} \left(1 - \alpha^2(x_2,t)\right) \tag{69}$$

for some $\varepsilon \in [0,1)$. Since $\mathcal{U} = \{|\alpha| < 1\}$, (56) holds. Therefore Theorem 6.2 is applicable.

Generic initial data; Theorem 2.5.

It is also possible to use the convex integration scheme to construct (generic) wild initial data for a given $\bar{e}$. Here we *first* fix $\bar{e} \in C([0,T] \times \mathbb{T}^n)$ and *then* construct the subsolution. The following is essentially from [DLS10] and [SW12].

In line with the strategy from Section 5.3, define the space of *initial data subordinate to* $\bar{e}$ as

$$Y_0 = \Big\{ w \in C^\infty(\mathbb{T}^n) : \operatorname{div} w = 0 \text{ and } \exists\, (\bar{v},\bar{u},\bar{q}) \in C(\mathbb{T}^n \times [0,T)) \text{ such that}$$
$$\text{(i) (65) holds with } \bar{v}|_{t=0} = w,$$
$$\text{(ii) } e(\bar{v},\bar{u}) < \bar{e} \text{ in } \mathbb{T}^n \times [0,T) \Big\}.$$

Observe that in particular

$$\tfrac{1}{2}|w(x)|^2 \le e(\bar{v}(x,0),\bar{u}(x,0)) < \bar{e}(x,0) \text{ in } \mathbb{T}^n \text{ for any } w \in Y_0.$$

We start with the following variant of Lemma 5.5, see [DLS10]:

Lemma 6.6. *There exists a convex, monotone increasing function*

$$\Phi : \mathbb{R} \to \mathbb{R} \quad \textit{with } \Phi(0) = 0 \textit{ and } \Phi(\tau) > 0 \textit{ for } \tau > 0 \tag{70}$$

with the following property: For any $w \in Y_0$ there exists a sequence $w_k \in Y_0$ such that $w_k \rightharpoonup w$ in $L^2(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n} |w_k - w|^2 \, dx \ge \Phi\left(\int_{\mathbb{T}^n} \bar{e}(x,0) - \tfrac{1}{2}|w|^2 \, dx \right)$$

and moreover, the corresponding subsolutions $(\bar{v}_k,\bar{u}_k,\bar{q}_k)$ in the definition of Y_0 can be chosen so that

$$\operatorname{supp}(\bar{v}_k - \bar{v}, \bar{u}_k - \bar{u}, \bar{q}_k - \bar{q}) \subset \mathbb{T}^n \times [0, 1/k).$$

Proof. Let $\bar{e} > 0$.

Step 1. First of all we claim that there exists a convex monotone function Φ as in (70) such that for any $(\bar{v},\bar{u}) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n}$ with $e(\bar{v},\bar{u}) < \bar{e}$ there exists $(\hat{v},\hat{u})$ with $e(\bar{v} \pm \hat{v}, \bar{u} \pm \hat{u}) < \bar{e}$ and

$$|\hat{v}|^2 \ge 4\,\Phi(|\bar{v}|^2 - 2\bar{e})$$

Indeed, this is a purely geometric fact concerning convex hulls, see (H2), and Lemma 5.3.

Step 2. Using the considerations in Section 6.1 concerning the conditions (H1) and (H2) we deduce: for any $(\bar{v},\bar{u}) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n}$ with $e(\bar{v},\bar{u}) < \bar{e}$ there exists $\delta > 0$ and a sequence $(v_k,u_k,q_k) \in C_c^\infty(B_1(0) \times (0,1))$ such that

(a) (53) holds;
(b) $e\big(\bar{v} + v_k(x,t), \bar{u} + u_k(x,t)\big) < \bar{e} - \delta$ for all (x,t);
(c) $v_k(\cdot,0) \rightharpoonup 0$ weakly in $L^2(\mathbb{T}^n)$;

(d) $\int_{B_1(0)} |v_k(x,0)|^2 dx > 2\,\Phi\big(\bar e - \frac12|\bar v|^2\big)$.

Notice that (c) and (d) involve integrals *in space only.* To see that this can be achieved, recall from (H1) (c.f. (61) and (64)) that v_k has the form at time $t=0$

$$v_k(x,0) = \hat v \sin(kx\cdot\xi) + O\left(\frac{1}{k}\right).$$

Step 3. Let $\bar e \in C(\mathbb{T}^n\times[0,T)$ and $w\in Y_0$ with associated strict subsolution $(\bar v,\bar u,\bar q)$. Using the continuity of $\bar e$ and $(\bar v,\bar u)$ we can now proceed as in the proof of Lemma 5.5as follows.

Given $x_0\in\mathbb{T}^n$ and $\varepsilon, r_0>0$, we apply Step 1 with $(\bar v,\bar u)(x_0,0)$ to obtain a sequence $(v_k,u_k,q_k)\in C_c^\infty(B_{r_0}(x_0)\times[0,\varepsilon))$ such that (a)-(c) holds and (d) is replaced by

(d') $\int_{B_{r_0}(x_0)} |v_k(x,0)|^2 dx > 2\,|B_{r_0}(x_0)|\Phi\big(\bar e - \frac12|\bar v|^2\big)$.

Moreover, we may choose $r_0>0$ so that (b) can be replaced by

(b') $e\big(\bar v(x,t)+v_k(x,t), \bar u(x,t)+u_k(x,t)\big) < \bar e$ for all (x,t);

We then find a sequence of disjoint balls $B_i = B_{r_i}(x_i)$, $i=1,\dots,I$ and associated sequences (v_k^i,u_k^i,q_k^i) such that (a),(b'),(c),(d') hold for each i,k,

$$\operatorname{supp}(v_k^i-\bar v, u_k^i-\bar u, q_k^i-\bar q)\subset\mathbb{T}^n\times[0,1/k)$$

and

$$\int_{\mathbb{T}^n}\Phi\Big(\bar e(x,0)-\tfrac12|\bar v(x,0)|^2\Big)\,dx \le 2\sum_{i=1}^{I}|B_i|\Phi\Big(\bar e(x_i,0)-\tfrac12|\bar v(x_i,0)|^2\Big)$$

We set

$$\begin{aligned}(\bar v_k,\bar u_k\bar q_k) &:= (\bar v,\bar u_k,\bar q_k)+\sum_{i=1}^{I}(v_k^i,u_k^i,q_k^i)\,,\\ w_k(x) &:= \bar v_k(x,0).\end{aligned}$$

By construction $w_k\in Y_0$ and $w_k\rightharpoonup w$ in $L^2(\mathbb{T}^n)$. Moreover

$$\int_{\mathbb{T}^n}|w_k-w|^2\,dx \ge \Phi\left(\int_{\mathbb{T}^n}\bar e(x,0)-\tfrac12|w|^2\,dx\right)$$

as required. □

As in Section 4 the perturbation property proved in Lemma 6.6 above leads to the following conditional existence statement.

Corollary 6.7. *For any $w\in Y_0$ there exists $\bar w\in L^2(\mathbb{T}^n)$ and a subsolution $(\bar v,\bar u,\bar q)\in C(\mathbb{T}^n\times(0,T))$ such that (65) holds with $\bar v|_{t=0}=\bar w$, $e(\bar v,\bar u)<\bar e$ in $\mathbb{T}^n\times(0,T)$,*

$$\tfrac12|\bar w(x)|^2 = \bar e(x,0)\quad \textit{for a.e. } x\in\mathbb{T}^n$$

and moreover

$$\|\bar w-w\|^2_{L^2(\mathbb{T}^n)} \le 3\int_{\mathbb{T}^n}\bar e(x,0)-\tfrac12|w(x)|^2\,dx.$$

Proof. Let $\delta > 0$. We proceed as in the proof of Theorem 4.2. Set $w_0 = w$ and $(\bar v_0, \bar u_0, \bar q_0)$ the associated subsolution. Having obtained w_k and $(\bar v_k, \bar u_k, \bar q_k)$, set

$$\Phi_k := \Phi\left(\int_{\mathbb{T}^n} \bar e(x,0) - \tfrac12 |w_k|^2\,dx\right).$$

Using Lemma 6.6 above, choose $w_{k+1} \in Y_0$ and associated $(\bar v_{k+1}, \bar u_{k+1}, \bar q_{k+1})$ such that

$$|\langle w_{k+1} - w_k, w_l\rangle| \le \min\left\{\delta 2^{-k}, \tfrac12\Phi_k\right\} \text{ for all } l \le k, \tag{71}$$

$$\int_{\mathbb{T}^n} |w_{k+1} - w_k|^2\,dx \ge \Phi_k, \tag{72}$$

$$\operatorname{supp}(\bar v_{k+1} - \bar v_k, \bar u_{k+1} - \bar u_k, \bar q_{k+1} - \bar q_k) \subset \mathbb{T}^n \times [0, 1/k), \tag{73}$$

where we write $\langle f, g\rangle = \int_{\mathbb{T}^n} fg\,dx$ for the L^2 inner product. From (71) and (72) we deduce that

$$\begin{aligned}\|w_{k+1}\|_2^2 &= \|w_k\|_2^2 + \|w_{k+1} - w_k\|_2^2 + 2\langle w_{k+1} - w_k, w_k\rangle\\ &\ge \|w_k\|_2^2 + \Phi_k - \Phi_k = \|w_k\|_2^2,\end{aligned}$$

so that the sequence $\|w_k\|_2$ is monotonic increasing. As $\|w_k\|_2^2 \le 2\int \bar e(x,0)\,dx < \infty$, it follows that $\|w_k\|_2$ is a convergent sequence. But then, as in the proof of Theorem 4.2, we deduce from (71) that $\{w_k\}$ is a Cauchy sequence in $L^2(\mathbb{T}^n)$, hence $w_k \to \bar w$. Using the condition on the supports (73), it is moreover easy to see that there exists $(\bar v, \bar u, \bar q) \in C(\mathbb{T}^n \times (0,T))$ such that

$$(\bar v_k, \bar u_k, \bar q_k) \to (\bar v, \bar u, \bar q) \text{ in } C_{loc}(\mathbb{T}^n \times (0,T)).$$

Consequently $(\bar v, \bar u, \bar q)$ also satisfies (65) and $\bar v|_{t=0} = \bar w$ in the weak L^2 sense. Finally, (72) implies that $\frac12|\bar w(x)|^2 = \bar e(x,0)$ for a.e. $x \in \mathbb{T}^n$. Furthermore, using once more (71) we obtain

$$|\langle \bar w - w, w\rangle| \le \delta$$

so that

$$\begin{aligned}\|\bar w - w\|_2^2 &= \|\bar w\|_2^2 - \|w\|_2^2 - 2\langle \bar w - w, w\rangle\\ &\le 2\int_{\mathbb{T}^n} \bar e(x,0) - \tfrac12|w(x)|^2\,dx + 2\delta\\ &\le 3\int_{\mathbb{T}^n} \bar e(x,0) - \tfrac12|w(x)|^2\,dx\end{aligned}$$

for $\delta > 0$ sufficiently small. This concludes the proof. $\square$

In the next lemma we show how to obtain smooth subsolutions from Leray weak solutions of the Navier-Stokes equations. See [SW12], where also the more general problem of obtaining regularized measure-valued solutions is considered.

Lemma 6.8. *Let $w \in L^2(\mathbb{T}^n)$ with $\operatorname{div} w = 0$. For any $\delta > 0$ there exists $(\bar v, \bar u, \bar q) \in C^\infty(\mathbb{T}^n \times [0,T))$ so that (53) holds,*

$$\left\|\bar v|_{t=0} - w\right\|_{L^2(\mathbb{T}^n)}^2 \le \delta \tag{74}$$

and for all $t \in [0,T]$

$$\int_{\mathbb{T}^n} e\big(\bar v(x,t), \bar u(x,t)\big)\,dx \le \tfrac12\int_{\mathbb{T}^n} |w(x)|^2\,dx + \delta\,. \tag{75}$$

Proof. By regularizing w if necessary, we may assume without loss of generality that $w \in C^\infty(\mathbb{T}^n)$.

For any $\nu > 0$ there exists a Leray weak solution v_ν of the Navier-Stokes system with viscosity $\nu > 0$

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= \nu \Delta v, \\ \operatorname{div} v &= 0, \\ v|_{t=0} &= w, \end{aligned}$$

together with the uniform bound

$$\sup_t \|v_\nu(t)\|^2_{L^2(\mathbb{T}^n)} + 2\nu \int_0^T \|\nabla v_\nu(t)\|^2_{L^2(\mathbb{T}^n)}\, dt \le \|w\|^2_{L^2(\mathbb{T}^n)}. \tag{76}$$

Since $\Delta v = \operatorname{div}(\nabla v + \nabla v^T)$ for div-free fields, we can define u_ν and q_ν as

$$\begin{aligned} u_\nu &= v_\nu \otimes v_\nu - \tfrac{1}{n}|v_\nu|^2 I - \nu(\nabla v_\nu + \nabla v_\nu^T), \\ q_\nu &= p_\nu + \tfrac{1}{n}|v_\nu|^2\,, \end{aligned}$$

so that u_ν takes values in $\mathcal{S}_0^{n\times n}$ and (v_ν, u_ν, q_ν) satisfies (53). Next, let $\rho(x) \in C_c^\infty(B_1(0))$ and $\chi \in C_c^\infty(-1,0)$ be standard mollifying kernels. fix $\varepsilon > 0$ and let $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(\varepsilon^{-1}x)$, $\chi_\varepsilon(t) = \varepsilon^{-1}\chi(\varepsilon^{-1}t)$. Set

$$\bar v(x,t) := \overline{v_\nu}(x,t) = \int_t^{t+\varepsilon} (v_\nu * \rho_\varepsilon)(x,s)\chi_\varepsilon(t-s)\, ds,$$

and similarly define $\bar u$ and $\bar q$, so that

$$\bar u = \overline{v_\nu \otimes v_\nu} - \tfrac{1}{n}\overline{|v_\nu|^2} I - \nu(\nabla \bar v + \nabla \bar v^T).$$

By virtue of the linearity of (53), the triple $(\bar v, \bar u, \bar q) \in C^\infty(\mathbb{T}^n \times [0,T))$ satisfies (53). Using Jensen's inequality we see that

$$\overline{v_\nu \otimes v_\nu} - \overline{v_\nu} \otimes \overline{v_\nu} \ge 0,$$

and hence

$$\bar v \otimes \bar v - \bar u \le \tfrac{1}{n}\overline{|v_\nu|^2} I + O\Big(\nu |\nabla \bar v|\Big).$$

Consequently, by integrating in x and using the uniform bound (76),

$$\int_{\mathbb{T}^n} e\big(\bar v(x,t), \bar u(x,t)\big)\, dx \le \int_{\mathbb{T}^n} |w(x)|^2\, dx + O\Big(\frac{\nu}{\varepsilon}\Big).$$

Finally,

$$\bar v|_{t=0} - w = \int_0^\varepsilon ((v_\nu - w) * \rho_\varepsilon)(s)\chi_\varepsilon(-s)\, ds + (w * \rho_\varepsilon - w),$$

so that

$$\|\bar v|_{t=0} - w\|_{L^2(\mathbb{T}^n)} \le \sup_{t\in[0,\varepsilon]} \|v_\nu(\cdot,t) - w\|_{L^2(\mathbb{T}^n)} + \|w * \rho_\varepsilon - w\|_{L^2(\mathbb{T}^n)}\,.$$

Once more using (76) and (53), and writing $v_\nu(t) := v_\nu(\cdot, t)$, we have

$$\begin{aligned}\int_{\mathbb{T}^n} |v_\nu(t) - w|^2\, dx &= \int_{\mathbb{T}^n} |v_\nu(t)|^2 - |w|^2\, dx - 2\int_{\mathbb{T}^n} (v_\nu(t) - w)\cdot w\, dx \\ &\le -2\int_{\mathbb{T}^n} (v_\nu(t) - w)\cdot w\, dx \\ &= 2\int_0^t \int_{\mathbb{T}^n} \Big(\operatorname{div} u_\nu(s) + \nabla q_\nu(s)\Big)\cdot w\, dxds \\ &= 2\int_0^t \int_{\mathbb{T}^n} u_\nu(s)\cdot \nabla w\, dxds \\ &\le C(t + t^{1/2})\|\nabla w\|_{L^\infty(\mathbb{T}^n)}\|w\|_{L^2(\mathbb{T}^n)}\,.\end{aligned}$$

Therefore, by choosing $\varepsilon > 0$ sufficiently small, we can ensure (74), and then by choosing $\nu > 0$ sufficiently small, we ensure in addition (75). The lemma follows. $\square$

Finally, we are ready to prove Theorem 2.5. Let $w \in L^2(\mathbb{T}^n)$ with $\operatorname{div} w = 0$ and let $\delta > 0$. Using Lemma 6.8 we obtain a smooth subsolution (v, u, q) satisfying (74)-(75). Choose $\bar{e} \in C(\mathbb{T}^n \times [0, T))$ so that $\bar{e} > e(v, u)$ in $\mathbb{T}^n \times [0, T)$ and

$$\int_{\mathbb{T}^n} \bar{e}(x, t) - e(v(x, t), u(x, t))\, dx \le \delta \qquad \text{for all } t \in [0, T).$$

Then, by construction, $v|_{t=0} \in Y_0$, so that, by Corollary 6.7 we obtain a $w \in L^2(\mathbb{T}^n)$ and associated $(\bar{v}, \bar{u}, \bar{q}) \in C(\mathbb{T}^n \times (0, T))$ so that

$$e(\bar{v}, \bar{u}) < \bar{e} \quad \text{ in } \mathbb{T}^n \times (0, T), \tfrac{1}{2}|\bar{w}|^2 \qquad\qquad = \bar{e}(\cdot, 0).$$

Therefore we can apply Theorem 6.2 with $\mathscr{U} = \mathbb{T}^n \times (0, T)$. In particular $\bar{w}$ is a wild initial data. Furthermore

$$\begin{aligned}\|\bar{w} - w\|_{L^2}^2 &\le 2\|\bar{w} - v|_{t=0}\|_{L^2}^2 + 2\|v|_{t=0} - w\|_{L^2}^2 \\ &\le C\int_{\mathbb{T}^n} \bar{e}(x, 0) - \tfrac{1}{2}|v(x, 0)|^2\, dx + C\delta \\ &\le C\int_{\mathbb{T}^n} e(v(x, 0), u(x, 0)) - \tfrac{1}{2}|v(x, 0)|^2\, dx + C\delta \\ &\le C\int_{\mathbb{T}^n} |w(x)|^2 - |v(x, 0)|^2\, dx + C\delta \\ &\le C\delta.\end{aligned}$$

Since $\delta > 0$ was arbitrary, this proves the strong density of wild initial data, Theorem 2.5.

7. Hölder regularity

In the previous sections we gave several examples where a very large set of solutions with low regularity exists. In particular

- C^1 isometric embeddings $M^n \hookrightarrow \mathbb{R}^{n+1}$;
- L^∞ weak solutions of the Euler equations.

In both cases it is natural to ask whether solutions with better regularity exist, and indeed, in many cases the answer is yes. However, under better regularity assumptions new conserved quantities appear, leading to rigidity phenomena that replace the type of flexibility (h-principle) of the irregular case. Thus

(a) the C^2 isometric embedding $S^2 \hookrightarrow \mathbb{R}^3$ is unique (modulo rigid motion);
(b) a Lipschitz solution of the initial value problem for Euler is unique (if it exists).

Observe that in both cases the existence of one more derivative is assumed. There are also more subtle ways in which a certain form of "rigidity" can appear:

(c) for $C^{1,\alpha}$ isometric embeddings with $\alpha > 1/2$ the holonomy group is preserved by the embedding;
(d) a $C^{1,\alpha}$ isometric embedding $S^2 \hookrightarrow \mathbb{R}^3$ with $\alpha > 2/3$ agrees with the standard embedding;
(e) for C^α weak solutions of Euler with $\alpha > 1/3$ the energy is constant in time.

7.1. **Standard inequalities.** The Hölder seminorms for $m = 0, 1, 2\dots$ and $\alpha \in (0,1)$ are defined as

$$[f]_m = \max_{|\beta|=m} \|D^\beta f\|_0\,,$$
$$[f]_{m+\alpha} = \max_{|\beta|=m} \sup_{x\neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}\,,$$

where $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ is a multi-index and $|\beta| = \beta_1 + \dots + \beta_n$. The Hölder norms are then given by

$$\begin{aligned}\|f\|_m &= \sum_{j=0}^m [f]_j\\ \|f\|_{m+\alpha} &= \|f\|_m + [f]_{m+\alpha}.\end{aligned}$$

The following inequalities are standard and can be found for instance in [Kry96]:

Lemma 7.1. *Let $f, g \in C^r(\mathbb{R}^n)$. For any $0 \le s \le r$ we have*

$$[f]_s \le C_r\big([f]_r + \|f\|_0\big) \tag{77}$$

$$[f]_s \le C_r \|f\|_0^{1-s/r} [f]_r^{s/r} \tag{78}$$

$$[fg]_r \le C_r\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) \tag{79}$$

Next, we record a couple of estimates on convolutions. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ such that

$$\rho \ge 0, \quad \rho(x) = \rho(-x), \quad \int \rho\, dx = 1, \tag{80}$$

and let $\rho_\ell(x) = \ell^{-n}\rho(\ell^{-1}x)$ for $0 < \ell < 1$. The first two estimates below are standard, the third one is from [CET94], see also [CDLS12].

Lemma 7.2.

$$\|f * \rho_\ell\|_r \le C_r \ell^{-r} \|f\|_0 \quad \textit{for } r \ge 0, \tag{81}$$

$$\|f * \rho_\ell - f\|_0 \le C_r \ell^r \|f\|_r \quad \textit{for } r \le 2, \tag{82}$$

$$\|(fg) * \rho_\ell - (f * \rho_\ell)(g * \rho_\ell)\|_r \le C_r \ell^{2s-r} [f]_s [g]_s \quad \textit{for } r \ge 0,\ s \le 1. \tag{83}$$

Proof. It suffices to prove the inequalities for $r \in \mathbb{N}$, the non-integer cases follow by interpolation (c.f. Lemma 7.1). For any multi-index β we have

$$\partial^\beta (f * \rho_\ell) = \ell^{-n-|\beta|} \int_{\mathbb{R}^n} (\partial^\beta \rho) \left(\frac{x-y}{\ell} \right) f(y)\, dy.$$

Hence

$$[f * \rho_\ell]_r \le \ell^{-r} \|f\|_0 \sum_{|\beta|=r} \int |\partial^\beta \rho|\, dx,$$

proving (81). Conversely,

$$f(x) - f * \rho_\ell(x) = \int_{\mathbb{R}^n} \rho_\ell(x-y)(f(y) - f(x))\, dy,$$

and by the mean value theorem $|f(y) - f(x)| \le [f]_1 |y-x|$, so that

$$|f(x) - f * \rho_\ell(x)| \le \ell [f]_1 \int |z \rho(z)|\, dz.$$

More generally, by considering the Taylor expansion of f we have $f(y) - f(x) = \sum_i \partial_i f(x)(y_i - x_i) + r_x(y-x)$, where $\sup_x |r_x(z)| \le C|z|^2 [f]_2$. Since ρ is symmetric,

$$\int_{\mathbb{R}^n} \rho(z) \sum_i \partial_i f(x) z_i\, dz = \sum_i \partial_i f(x) \int_{\mathbb{R}^n} \rho(z) z_i\, dz = 0,$$

so that

$$f(x) - f * \rho_\ell(x) = - \int_{\mathbb{R}^n} \rho_\ell(z) r_x(z)\, dz.$$

This implies (82).

For the proof of estimate (83) let β be any multi-index with $|\beta| = r$. By the product rule

$$\begin{aligned}
\partial^\beta \big[\rho_\ell * (fg) - &(\rho_\ell * f)(\rho_\ell * g)\big] = \\
&= \partial^\beta \rho_\ell * (fg) - \sum_{\beta' \le \beta} \binom{\beta}{\beta'} (\partial^{\beta'} \rho_\ell * f)(\partial^{\beta-\beta'} \rho_\ell * g) \\
&= \partial^\beta \rho_\ell * (fg) - (\partial^\beta \rho_\ell * f)(\rho_\ell * g) + (\rho_\ell * f)(\partial^a \rho_\ell * g) \\
&\quad - \sum_{0<\beta'<\beta} \binom{\beta}{\beta'} [\partial^{\beta'} \rho_\ell * (f - f(x))][\partial^{\beta-\beta'} \rho_\ell * (g - g(x))] \\
&= \partial^\beta \rho_\ell * [(f - f(x))(g - g(x))] \\
&\quad - \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \partial^{\beta'} \rho_\ell * (f - f(x)) \cdot \partial^{\beta-\beta'} \rho_\ell * (g - g(x)),
\end{aligned} \tag{84}$$

where we have used the fact that

$$\partial^\beta \rho_\ell * f(x) = \begin{cases} f(x) & \text{if } \beta = 0, \\ 0 & \text{if } \beta \ne 0. \end{cases}$$

Now observe that for $s \leq 1$

$$
\begin{aligned}
&\left|\partial^\beta \rho_\ell * [(f - f(x))(g - g(x))]\right| \\
&\qquad = \left|\int \partial^\beta \rho_\ell(y)(f(x-y) - f(x))(g(x-y) - g(x))dy\right| \\
&\qquad \leq \int |\partial^\beta \rho_\ell(y)||y|^{2s} dy\, [f]_s [g]_s \;=\; C_r\, \ell^{2s-r}[f]_s[g]_s.
\end{aligned}
$$

Similarly, all the terms in the sum over β' in (84) obey the same estimate. This concludes the proof of (83). □

7.2. **Conserved quantities.**
Euler flows and Energy.

The following proof is essentially from [CET94], see also [DR00b]. Let $v \in L^\infty(0, T; C^\alpha(\mathbb{T}^3))$ be a weak solution of the Euler equations for some $\alpha > 0$. Since v is not sufficiently regular, the identities (6) and (7) cannot be obtained by simply testing the equation with v itself. Instead we mollify the equation, obtaining

$$\partial_t(v * \rho_\ell) + \operatorname{div}\left[(v \otimes v) * \rho_\ell\right] + \nabla(p * \rho_\ell) = 0.$$

Observe that $v_\ell := v * \rho_\ell$ is not a solution of the Euler equations. However, testing with v_ℓ itself we obtain the following analogue of (6):

$$\partial_t \frac{|v_\ell|^2}{2} + \operatorname{div}\left(\left(\frac{|v_\ell|^2}{2} + p_\ell\right) v_\ell\right) = v_\ell \cdot \operatorname{div}\left[v_\ell \otimes v_\ell - (v \otimes v) * \rho_\ell\right]$$

which, after integrating, yields

$$\frac{d}{dt}\int_{\mathbb{T}^3} \frac{|v_\ell|^2}{2}\, dx = \int_{\mathbb{T}^3} \nabla v_\ell : \left[v_\ell \otimes v_\ell - (v \otimes v) * \rho_\ell\right] dx.$$

The integrand on the right can be estimated, using (81) and (83), as

$$\|\nabla v_\ell : \left[v_\ell \otimes v_\ell - (v \otimes v) * \rho_\ell\right]\|_0 \leq \|v_\ell\|_1 \|v_\ell \otimes v_\ell - (v \otimes v) * \rho_\ell\|_0 \leq C\ell^{3\alpha-1}[v]_\alpha^3,$$

so that, by letting $\ell \to 0$, we deduce that

$$\text{if } \alpha > 1/3, \text{ then} \qquad \frac{d}{dt}\int_{\mathbb{T}^3} \frac{|v|^2}{2}\, dx = 0.$$

Isometric immersions and Curvature.

Let g be a C^2 metric on $\Omega \subset \mathbb{R}^n$ and $u \in C^{1,\alpha}(\Omega; \mathbb{R}^m)$ an immersion for some $\alpha > 0$. Recall that on the manifold (Ω, g) one defines the Christoffel symbols and the Riemannian curvature tensor as

$$\Gamma^i_{jk} = \sum_l \frac{1}{2} g^{il}\left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj}\right)$$

and

$$R_{iljk} = \sum_p g_{lp}\left(\partial_k \Gamma^p_{ij} - \partial_j \Gamma^p_{ik} + \Gamma^l_{ij}\Gamma^p_{kl} - \Gamma^l_{ik}\Gamma^p_{jl}\right).$$

In particular for $n = 2$ the Gauss curvature can be calculated as

$$\kappa = \frac{R_{1212}}{\det g}.$$

It is important to observe that, although u is not regular enough to take more than one derivative, $g = \nabla u^T \nabla u$ is. As in the case of the Euler equations, let us mollify u, i.e. set

$$u_\ell = u * \rho_\ell, \qquad g_\ell = \nabla u_\ell^T \nabla u_\ell.$$

Using (82) we see that

$$\|u_\ell - u\|_1 \le C\ell^\alpha \|\nabla u\|_\alpha, \tag{85}$$

and (83) yields

$$\begin{aligned}\|g_\ell - g\|_r &\le \|\nabla u_\ell^T \nabla u_\ell - (\nabla u^T \nabla u) * \rho_\ell\|_r + \|g * \rho_\ell - g\|_r \\ &\le C_r \ell^{2\alpha - r} \|u\|_\alpha^2 + C_r \ell^{2-r} \|g\|_2.\end{aligned}$$

In particular, if we set Γ_ℓ to be the Christoffel symbol associated with the metric g_ℓ, then

$$\|\Gamma_\ell - \Gamma\|_0 \le C\|g_\ell - g\|_1 \le C\ell^{2\alpha-1}. \tag{86}$$

Thus, provided $\alpha > 1/2$, the Christoffel symbols converge uniformly. Consequently, parallel transport can be defined on the image $u(\Omega)$ by regularization, see [Bor59] and (c) above in the introduction to Section 7.

Concerning the curvature, recall the following classical formula of Gauss, relating the Gauss curvature of an embedded (smooth) surface Σ^2 to the Gauss map:

$$\kappa \, dA = N^* d\sigma,$$

where dA is the area form on the surface, $d\sigma$ is the area form on S^2 and $N : \Sigma^2 \to S^2$ is the unit normal to $\Sigma^2 \subset \mathbb{R}^3$, the Gauss map.

Returning to isometric embeddings, let M^2 be a compact surface without boundary, and $u : M^2 \to \mathbb{R}^3$ an isometric embedding of class $C^{1,\alpha}$. Let u_ℓ once again be the regularized embedding. Using the area formual, we have the equivalent formulation of the above as

$$\int_V f(N_\ell(x))\kappa_\ell(x)\, dA_\ell(x) = \int_{S^2} f(y) \deg(y, V, N_\ell)\, d\sigma(y) \tag{87}$$

for all open $V \subset\subset \Sigma^2$ and $f \in C^1(S^2)$. From (85) we deduce that $N_\ell \to N$ uniformly, provided $\alpha > 0$, and hence the right hand side of (87) converges to $\int_{S^2} f(y) \deg(y, V, N)\, d\sigma(y)$.

Concerning the left hand side, assume now that $\alpha > 2/3$. Observe from the formulas for the Riemannian curvature tensor and for κ, that, by (86), only the term with a second derivative on g_ℓ remains problematic. Accordingly, let us consider a term of the form

$$\int_V f(N_\ell(x))\partial_{kl} g_\ell(x)\, (\det g_\ell)^{-1/2} dx. \tag{88}$$

Set $\psi_\ell := f(N_\ell)(\det g_\ell)^{-1/2}$, so that (88) can be written as

$$\begin{aligned}\int_V \psi_\ell \partial_{kl} g_\ell &= -\int_V \partial_k \psi_\ell \partial_l g_\ell \\ &= -\int_V \partial_k \psi_\ell (\partial_l g_\ell - \partial_l g)\, dx - \int_V \partial_k \psi_\ell \partial_l g\, dx \\ &= -\int_V \partial_k \psi_\ell (\partial_l g_\ell - \partial_l g)\, dx + \int_V \psi_\ell \partial_{kl} g\, dx = I_\ell + II_\ell.\end{aligned}$$

Using the estimate on $g_\ell - g$ and (81), we have

$$\|\psi_\ell(\partial_l g_\ell - \partial_l g)\|_0 \leq C\ell^{\alpha-1}\ell^{2\alpha-1} = C\ell^{3\alpha-2},$$

leading to $I_\ell \to 0$, whereas $II_\ell \to II = \int_V \psi\partial_{kl}g\,dx$. Consequently, if $\alpha > 2/3$, the Gauss formula in the area formulation (87) holds for the surface $\Sigma^2 = u(M^2)$.

The rigidity statement (d) from the introduction of this section follows from the following notion from [Pog73]:

Definition 7.3. *Let $\Omega \subset \mathbb{R}^2$ be open and $u \in C^1(\Omega;\mathbb{R}^3)$ an immersion. The surface $u(\Omega)$ has* bounded extrinsic curvature *if there exists a constant C such that*

$$\sum_{i=1}^{k}\big|N(E_i)\big| \leq C$$

for any finite collection $\{E_i\}$ of pairwise disjoint closed subsets of Ω.

The rigidity for surfaces with bounded extrinsic curvature follows from the classical rigidity theory for convex surfaces (see e.g. [CV27, Pog51, Spi79] and

Theorem 7.4 (Pogorelov [Pog73])**.** *Closed C^1 surfaces with positive Gauss curvature and bounded extrinsic curvature are convex.*

It remains to show therefore, that our surface $u(M^2)$ with $u \in C^{1,\alpha}$, $\alpha > 2/3$ has indeed bounded extrinsic curvature.

Lemma 7.5. *Let $\Sigma^2 = u(M^2)$ be a compact surface with positive Gauss curvature and assume that the area formula*

$$\int_V f(N(x))\kappa(x)\,dA(x) = \int_{S^2} f(y)\deg(y,V,N)\,d\sigma(y)$$

holds for all open $V \subset\subset \Sigma^2$ and $f \in C^1(S^2)$. Assume further that $u \in C^{1,\alpha}$ with $\alpha > 1/2$. Then Σ^2 is of bounded extrinsic curvature.

Proof. Using the fact that $\kappa > 0$ and the area formula holds, it is not difficult to verify that

- $\deg(y,V,N) \geq 0$ for all $y \in S^2$;
- $\deg(y,V,N) \geq 1$ for all $y \in N(V)\setminus N(\partial V)$.

For the details see [CDLS12]. Then, given a pairwise disjoint family of closed sets $\{E_i\}$, find an open cover with pairwise disjoint sets $\{V_i\}$ such that ∂V_i is smooth. Then

$$\begin{aligned}\sum_i\big|N(E_i)\big| &\leq \sum_i\big|N(V_i)\setminus N(\partial V_i)\big| + \big|N(\partial V_i)\big| \\ &\leq \sum_i\int_{S^2}\deg(y,V_i,N)\,dy \\ &= \sum_i\int_{V_i}\kappa \leq \int_{M^2}\kappa < \infty,\end{aligned}$$

where we have used, that for $\alpha > 1/2$ the measure of the image $N(\partial V_i)$ is zero, since $|\partial V_i| = 0$ and $N \in C^\alpha$. This completes the proof. $\square$

7.3. $C^{1,\alpha}$ **isometric embeddings.** Let us return to the local codimension 2 setting of Section 3. Our aim is to modify the proof of Theorem 3.1 so that the embedding obtained has Hölder continuous first derivatives. The main idea can be best explained in the local, codimension 2 setting of Section 3.1. The extensions in Section 3.2 can then be implemented in the same way, but the details become more technical without more insight. Therefore, in these notes we restrict our attention to the setting of Theorem **??**. The general case can be found in [CDLS12].

The iteration scheme will proceed essentially in the same way as in Section 3. In particular, recall that one *step* was defined by

$$v(x) = u(x) + \frac{a(x)}{\lambda}\Big(\sin(\lambda x\cdot\xi)\zeta(x) + \cos(\lambda x\cdot\xi)\eta(x)\Big). \tag{89}$$

This time we need a more precise estimate on the $O(\frac{1}{\lambda})$-term in the metric error.

Lemma 7.6. *For v defined above*

$$\nabla v^T\nabla v = \nabla u^T\nabla u + a^2\,\xi\otimes\xi + E,$$

where

$$\|E\|_0 \leq \frac{C}{\lambda}\left(\|a\|_0\|a\|_1 + \|a\|_0\|u\|_2\right) + \frac{C}{\lambda^2}\left(\|a\|_1^2 + \|a\|_0^2\|u\|_2^2\right).$$

Moreover, for any $k\geq 1$ we have

$$[v-u]_{k+1} \leq C_k\left(\lambda^k\|a\|_0 + \frac{1}{\lambda}(\|a\|_{k+1} + \|u\|_{k+2})\right).$$

Proof. The proof is a direct calculation, using Lemma 7.1 and the estimate

$$\|\zeta\|_k,\ \|\eta\|_k \leq C_k\|u\|_{k+1}.$$

□

Observe the following effect: in order to estimate the new metric error, we need information on the second derivatives of u. However, in order to estimate the second derivatives of v (for the subsequent step), we need information on the third derivatives of u. In order words the iteration scheme comes with a *loss of derivative*, analogously to the Nash-Moser iteration scheme. The solution, as in the Nash-Moser iteration, is to mollify the maps at each stage of the iteration. Recall that a *stage* consists of decomposing the metric error

$$h = g - \nabla u^T\nabla u$$

into primitive metrics and then adding each primitive metric using the ansatz (89) in consecutive *steps*. This time we will mollify u at the beginning of each stage at length-scale $\ell > 0$ and estimate all norms in terms of

$$\delta^2 := \|h\|_0 \quad\text{and}\quad M := \|u\|_2.$$

Mollification.

Set

$$u_\ell := \rho_\ell * u, \quad g_\ell := \rho_\ell * g,$$

where ρ_ℓ is a standard mollifier as in (80). Since we aim at a local result, we may as well assume that u and g are defined in a neighbourhood of Ω. Using the convolution estimates in Lemma 7.2 we obtain

$$\begin{aligned} \|u_\ell\|_{k+2} &\leq C_k \ell^{-k} M \quad \text{ for } k \geq 0, \\ \|u_\ell - u\|_1 &\leq C \ell M, \end{aligned}$$

whereas the metric error becomes

$$\begin{aligned} \|g_\ell - \nabla u_\ell^T \nabla u_\ell\|_k &\leq \|g_\ell - (\nabla u^T \nabla u)_\ell\|_k + \|(\nabla u^T \nabla u)_\ell - \nabla u_\ell^T \nabla u_\ell\|_k \\ &\leq C_k (\ell^{-k}\delta^2 + \ell^{2-k} M^2). \end{aligned}$$

In order not to increase the size of the metric error by the mollification, we are forced to choose the length scale

$$\ell = \frac{\delta}{M}.$$

This in turn leads to the estimates ($k \geq 0$)

$$\begin{aligned} \|u_\ell - u\|_1 &\leq C\delta, \\ \|g_\ell - \nabla u_\ell^T \nabla u_\ell\|_k &\leq C_k \delta^2 \ell^{-k}, \\ \|u_\ell\|_{k+2} &\leq C_k \delta \ell^{-k-1}. \end{aligned}$$

Decomposing into primitive metrics.

We wish to use Lemma 3.3 to write the metric error

$$h_\ell := g_\ell - \nabla u_\ell^T \nabla u_\ell$$

as the (locally finite) sum of primitive metrics. However, the definition $a_j(x) = \Gamma_j(h_\ell(x))$ has the disadvantage that we cannot control higher derivatives of a_j easily. Recall that $\|a_j\|_0 \leq \|h_\ell\|_0^{1/2}$, so that, in analogy with the estimates for u_ℓ we should expect the estimates

$$\|a_j\|_k \leq C_k \delta \ell^{-k}. \tag{90}$$

Indeed, after mollification we can imagine g_ℓ, ∇u_ℓ and hence h_ℓ to be constant on length scales below ℓ. In order to achieve (90) we rescale the metric error as follows. Fix a small number $r > 0$ and define

$$\begin{aligned} \tilde{h}_\ell : &= g_\ell + \tfrac{r}{\delta^2} h_\ell \\ &= \left(1 + \tfrac{r}{\delta^2}\right) g_\ell - \tfrac{r}{\delta^2} \nabla u_\ell^T \nabla u_\ell \end{aligned}$$

and

$$\tilde{u}_\ell := \left(1 + \tfrac{\delta^2}{r}\right)^{-1/2} u_\ell.$$

Then

$$\|\tilde{h}_\ell - g_\ell\|_0 \leq r, \tag{91}$$

and on the other hand

$$g_\ell - \nabla \tilde{u}_\ell^T \nabla \tilde{u}_\ell = \frac{\delta^2}{r + \delta^2} \tilde{h}_\ell.$$

Invoking Lemma 3.3 we write

$$\tilde{h}_\ell(x) = \sum_j \Gamma_j^2(\tilde{h}_\ell(x))\xi^j \otimes \xi^j \tag{92}$$

and accordingly, we set

$$a_j(x) := \frac{\delta}{\sqrt{r+\delta^2}}\,\Gamma_j(\tilde{h}_\ell(x)). \tag{93}$$

Because of (91) and since the decomposition in Lemma 3.3 is locally finite, by choosing $r > 0$ sufficiently small (only depending on g), the sum (92) consists of a fixed finite number of terms $\{\Gamma_j\}_{j=1\dots N}$, where N is independent of u (hence independent of the particular stage). Therefore the functions Γ_j involved in the sum are smooth with uniform estimates on all derivatives (each single Γ_j has this property, see Lemma 3.3), and consequently

$$\begin{aligned} \|\Gamma_j \circ \tilde{h}_\ell\|_k &\le C_k \|\tilde{h}_\ell\|_k \\ &\le C_k \left(\|g_\ell\|_k + \tfrac{r}{\delta^2}\|h_\ell\|_k\right) \\ &\le C_k \ell^{-k}. \end{aligned} \tag{94}$$

From this together with (93) we deduce (90) easily.

The above rescaling can be understood as follows: The estimate in (94), $\|\Gamma_j \circ \tilde{h}_\ell\|_k \le C_{k,j}\|\tilde{h}_\ell\|_k$, holds for each j separately, but with constants that might depend on j as well. Moreover, along the iteration $\tilde{h}_\ell$ becomes smaller and smaller, requiring new Γ_j's to be used in (92). Indeed, recall that each Γ_j is supported on a compact subset of $\mathcal{P}$, the set of symmetric positive-definite matrices. On the other hand the identity (92) suggests that we might instead use a fixed set of Γ_j's if we extend them to 1/2-homogeneous functions. More precisely, Lemma 3.3 obviously implies the identity

$$A = \sum_j \left(|A|^{1/2}\Gamma_j\left(\frac{A}{|A|}\right)\right)^2 \xi^j \otimes \xi^j$$

for $A \in \mathcal{P}$. The estimate (91) ensures that only a finite number of Γ_j's are used, and (93) amounts to the 1/2-homogeneous rescaling, with $\delta \sim |h_\ell|$.

Next, observe that $\tilde{u}_\ell$ enjoys the same estimates as u_ℓ, since we may assume $\delta^2 < r$. ($r > 0$ is fixed once for all whereas $\delta > 0$ changes along the iteration).

Finally, we replace (89) by

$$v(x) = \tilde{u}_\ell(x) + \frac{a(x)}{\lambda}\Big(\sin(\lambda x\cdot\xi)\tilde{\zeta}_\ell(x) + \cos(\lambda x\cdot\xi)\tilde{\eta}_\ell(x)\Big), \tag{95}$$

where $\tilde{\zeta}_\ell, \tilde{\eta}_\ell$ are the unit normal vectors to $\tilde{u}_\ell(\Omega)$ and $a = a_j$, $\xi = \xi^j$ for some j. Combining Lemma 7.6 with the estimates for $\|u_\ell\|_{k+2}$ and $\|a_j\|_k$ from above, we obtain the new metric error:

$$\|E\|_0 \le \frac{C}{\lambda}(\delta^2\ell^{-1} + \delta M) + \frac{C}{\lambda^2}(\delta^2\ell^{-2} + \delta^2 M^2).$$

Since we want to reduce the metric error using this ansatz, at the least we should have $\|E\|_0 \le C\delta^2$. This requires

$$\lambda \ge \ell^{-1}.$$

Then, combining Lemma 7.6 with the estimates above and the choice $\ell = \frac{\delta}{M}$ we obtain

Lemma 7.7. *Let $\lambda \geq \ell^{-1}$. For v defined in (95) we have*

$$\nabla v^T \nabla v = \nabla \tilde{u}_\ell^T \nabla \tilde{u}_\ell + a^2 \, \xi \otimes \xi + E,$$

where

$$\|E\|_0 \leq C \frac{\delta M}{\lambda}.$$

Moreover, $\|v - u\|_1 \leq C\delta$ and for any $k \geq 1$ we have

$$\|v\|_{k+1} \leq C_k \lambda^k \delta.$$

The lemma above can now be iterated to obtain a stage, as in Proposition 3.4 from Section 3. The number of steps in a stage will be determined by the number of non-zero terms in the decomposition

$$g(x) = \sum_j \lambda^2(g(x)) \xi^j \otimes \xi^j \quad x \in \Omega.$$

Thus, let

$$n_* = \#\Big\{\lambda_j : \{\lambda_j > 0\} \cap g(\overline{\Omega}) \neq \emptyset\Big\}. \tag{96}$$

Proposition 7.8 (Stage)**.** *Let $m \geq n+2$ and $u : \Omega \to \mathbb{R}^m$ a smooth, strictly short immersion. There exists $\delta_0 > 0$ such that if $\|g - \nabla u^T \nabla u\|_0 < \delta_0^2$, then for any $K \geq 1$ there exists a smooth strictly short immersion $\tilde{u} : \Omega \to \mathbb{R}^m$ with*

$$\begin{aligned}
\|g - \nabla \tilde{u}^T \nabla \tilde{u}\|_0 &\leq C \frac{1}{K} \|g - \nabla u^T \nabla u\|_0, \\
\|\tilde{u} - u\|_1 &\leq C \|g - \nabla u^T \nabla u\|_0^{1/2}, \\
\|\tilde{u} - u\|_0 &\leq C \frac{1}{K} \|g - \nabla u^T \nabla u\|_0^2, \\
\|\tilde{u}\|_2 &\leq C \|u\|_2 \, K^{n_*}.
\end{aligned}$$

Proof. The proof proceeds analogously to the proof of Proposition 3.4, after some preparation. Set $\delta^2 := \|g - \nabla u^T \nabla u\|_0$ and $M := \|u\|_2$, as above, and $\ell := \frac{\delta}{M}$. Since $g(\overline{\Omega})$ is compact, n_* is finite. For $r > 0$ to be fixed, define $\tilde{h}_\ell$ as above. Using (91) we see that

$$\|g - \tilde{h}_\ell\|_0 \leq \|g - g_\ell\| + r.$$

Hence, by choosing $r > 0$ and $\delta_0 > 0$ (and consequently ℓ) sufficiently small, we can ensure that $\tilde{h}_\ell(\overline{\Omega})$ is covered by the same open sets $\{\lambda_j > 0\}$ as $g(\overline{\Omega})$. Consequently the decomposition (92) consists of fixed n_* terms (independently of u).

We define successively the maps

$$u_0 = \tilde{u}_\ell, \, u_1, u_2, \ldots, u_{n_*} = \tilde{u}$$

by applying Lemma 7.7 repeatedly with $\lambda_1, \ldots, \lambda_{n_*}$. From Lemma 7.7 we obtain inductively

$$\|u_{j+1}\|_2 \leq C \lambda_j \|g - \nabla u^T \nabla u\|_0^{1/2},$$

so that we require

$$\lambda_1 \geq \frac{K}{\ell} \quad \text{and} \quad \lambda_{j+1} \geq K \lambda_j,$$

leading to the choice of frequencies

$$\lambda_1 = \frac{K}{\ell}, \, \lambda_2 = \frac{K^2}{\ell}, \ldots, \lambda_{n_*} = \frac{K^{n_*}}{\ell}.$$

By summing up the errors obtained in each step we arrive at the claimed estimates for $\tilde{u}$. $\square$

Proposition 7.8 can now be easily iterated to obtain

Theorem 7.9. *Let $m \geq n+2$ and $u : \Omega \to \mathbb{R}^m$ a smooth, strictly short immersion. There exists $\delta_0 > 0$ such that if $\|g - \nabla u^T \nabla u\|_0 < \delta_0^2$, then for any $\varepsilon > 0$ there exists $\tilde{u} \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ for any*

$$\alpha < \frac{1}{1+2n_*}$$

such that $\|u - \tilde{u}\|_0 < \varepsilon$ and

$$\nabla \tilde{u}^T \nabla \tilde{u} = g \text{ in } \overline{\Omega}.$$

Proof. Let $u_0 = u$ and $M_0 = \|u\|_2$. By iterating over stages (Proposition 7.8) with some fixed $K > 1$ we obtain a sequence $u_k : \Omega \to \mathbb{R}^m$ with

$$\begin{aligned} \|g - \nabla u_k^T \nabla u_k\|_0 &\leq \delta_k^2, \\ \|u_k\|_2 &\leq M_k, \\ \|u_{k+1} - u_k\|_1 &\leq C\delta_k, \end{aligned}$$

where

$$\delta_{k+1}^2 = \frac{C}{K}\delta_k^2, \quad M_{k+1} = CM_k K^{n_*}.$$

By changing the value of $K > 1$ we may assume the constant is $C = 1$ in the relations above, so that

$$\delta_k = \delta_0 K^{-k/2}, \quad M_k = M_0 K^{kn_*}.$$

Therefore

$$\begin{aligned} &\|u_{k+1} - u_k\|_1 \leq C\delta_0 K^{-k/2} \\ &\|u_{k+1} - u_k\|_2 \leq M_0(K^{kn_*} + K^{(k+1)n_*}) = CK^{kn_*}, \end{aligned}$$

and by interpolation

$$\|u_{k+1} - u_k\|_{1+\alpha} \leq C\delta_0^{1-\alpha} K^{-\frac{k}{2}(1-\alpha(1+2n_*))}.$$

$\square$

Optimal Hölder exponent?

The optimal Hölder exponent is determined through n_*, the number of steps in a stage. In turn n_*, defined in (96), is determined entirely by the metric g. Recall also that λ_j are defined in Lemma 3.3 as convex coordinates on simplices $S \subset \mathcal{P}_1$, where $\mathcal{P}_1$ is the set of positive definite $n \times n$ matrices with trace 1. Thus, in the best case, if $g(\overline{\Omega})$ is contained in one simplex (e.g. if $g(x)$ is close to a fixed constant matrix for all $x \in \Omega$), then the number n_* is determined by Carathéodory's theorem as

$$n_* = \dim L + 1 = \tfrac{1}{2}n(n+1),$$

so that the best regularity obtainable via this method is

$$\alpha < \frac{1}{1+n+n^2}.$$

For $S^2 \hookrightarrow \mathbb{R}^3$ or more generally, for embedding surfaces, this yields $1/7$, and thus there remains a big gap in the range $1/7 < \alpha < 2/3$.

Observe also that in any case $n_* \geq 1$. Therefore, even if we were able to perform the steps in parallel rather than serially (e.g. in sufficiently high codimension this is possible), we would only be able to reach the Hölder exponent $\alpha < 1/3$. It turns out that in high codimension additional ideas can be used to improve on the scheme and in fact reach any exponent $\alpha < 1$, see [Käl78]. On the other hand, if the metric g is sufficiently smooth, by an entirely different construction, more based on Newton iteration rather than convex integration, much more smooth embeddings can be constructed. See [Nas56, Jac72].

References

[BDLS11] Y. Brenier, C. De Lellis, and L. Székelyhidi, Jr., *Weak-strong uniqueness for measure-valued solutions*, Comm. Math. Phys. **305** (2011), no. 2, 351–361. MR 2805464

[BGK00] C. Bardos, J.-M. Ghidaglia, and S. Kamvissis, *Weak convergence and deterministic approach to turbulent diffusion*, Nonlinear wave equations (Providence, RI, 1998), Contemp. Math., vol. 263, Amer. Math. Soc., 2000, pp. 1–15.

[Bor59] Ju. F. Borisov, *The parallel translation on a smooth surface. IV*, Vestnik Leningrad. Univ. **14** (1959), no. 13, 83–92. MR MR0112098 (22 #2956)

[Bor65] ———, *$C^{1,\alpha}$-isometric immersions of Riemannian spaces*, Doklady **163** (1965), 869–871.

[Bor04] ———, *Irregular $C^{1,\beta}$-surfaces with analytic metric.*, Sib. Mat. Zh. **45** (2004), no. 1, 25–61 (Russian, English).

[BT07] C. Bardos and E. Titi, *Euler equations for an ideal incompressible fluid*, Uspekhi Mat. Nauk **62** (2007), no. 3(375), 5–46. MR 2355417 (2008f:76017)

[CCFS08] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations*, Nonlinearity **21** (2008), no. 6, 1233–1252. MR 2422377 (2009g:76008)

[CDLS12] S. Conti, C. De Lellis, and L. Székelyhidi, Jr., *h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings*, Nonlinear Partial Differential Equations, Abel Symposia, vol. 7, Springer, 2012, pp. 83–116.

[CET94] P. Constantin, W. E, and E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. **165** (1994), no. 1, 207–209. MR MR1298949 (96e:76025)

[Cho94] A. J. Chorin, *Vorticity and turbulence*, Applied Mathematical Sciences, vol. 103, Springer-Verlag, New York, 1994. MR MR1281384 (95m:76043)

[CV27] St. Cohn-Vossen, *Zwei Sätze über die Starrheit der Eiflächen.*, Nachrichten Göttingen **1927** (1927), 125–137 (German).

[DiP85] R. J. DiPerna, *Compensated compactness and general systems of conservation laws*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 383–420. MR MR808729 (87g:35148)

[DLS09] C. De Lellis and L. Székelyhidi, Jr., *The Euler equations as a differential inclusion*, Ann. of Math. (2) **170** (2009), no. 3, 1417–1436. MR 2600877 (2011e:35287)

[DLS10] ———, *On admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal. **195** (2010), no. 1, 225–260. MR 2564474 (2011d:35386)

[DLS12a] ———, *Dissipative euler flows and onsager's conjecture*, Preprint. (2012), 1–40.

[DLS12b] ———, *The h-principle and the equations of fluid dynamics*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 3, 347–375.

[DLS13] ———, *Continuous dissipative Euler flows*, To appear in Inventiones (2013), 1–26.

[DM97] B. Dacorogna and P. Marcellini, *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases*, Acta Math. **178** (1997), 1–37. MR 98d:35029

[DR00a] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations*, Nonlinearity **13** (2000), no. 1, 249–255. MR 1734632 (2001c:76032)

[DR00b] ———, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations*, Nonlinearity **13** (2000), no. 1, 249–255. MR MR1734632 (2001c:76032)

[EM02] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, American Mathematical Society, Providence, RI, 2002. MR MR1909245 (2003g:53164)

[ES06] G. L. Eyink and K. R. Sreenivasan, *Onsager and the theory of hydrodynamic turbulence*, Rev. Modern Phys. **78** (2006), no. 1, 87–135. MR 2214822 (2007g:76108)

[Eyi94] G. L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer*, Phys. D **78** (1994), no. 3-4, 222–240. MR MR1302409 (95m:76020)

[FM99] I. Fonseca and S. Müller, *A-quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal. **30** (1999), no. 6, 1355–1390 (electronic). MR 1718306 (2000j:49020)

[Fri95] U. Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995, The legacy of A. N. Kolmogorov. MR MR1428905 (98e:76002)

[Gro86] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 9, Springer-Verlag, Berlin, 1986. MR 90a:58201

[Her43] G. Herglotz, *Über die Starrheit der Eiflächen.*, Abh. Math. Semin. Hansische Univ. **15** (1943), 127–129 (German).

[Jac72] H. Jacobowitz, *Implicit function theorems and isometric embeddings*, Ann. of Math. (2) **95** (1972), 191–225. MR MR0307127 (46 #6248)

[Käl78] A. Källén, *Isometric embedding of a smooth compact manifold with a metric of low regularity*, Ark. Mat. **16** (1978), no. 1, 29–50. MR MR499136 (80c:53064)

[Kir03] B. Kirchheim, *Rigidity and Geometry of microstructures*, Habilitation thesis, University of Leipzig, 2003.

[KMŠ03] B. Kirchheim, S. Müller, and V. Šverák, *Studying nonlinear PDE by geometry in matrix space*, Geometric analysis and Nonlinear partial differential equations (Stefan Hildebrandt and Hermann Karcher, eds.), Springer-Verlag, 2003, pp. 347–395.

[Kol91] A. N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*, Proc. Roy. Soc. London Ser. A **434** (1991), no. 1890, 9–13, Translated from the Russian by V. Levin, Turbulence and stochastic processes: Kolmogorov's ideas 50 years on. MR 1124922 (92h:76049)

[Kry96] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, vol. 12, American Mathematical Society, Providence, RI, 1996. MR 1406091 (97i:35001)

[Kui55] N. Kuiper, *On C^1 isometric imbeddings i,ii*, Proc. Kon. Acad. Wet. Amsterdam A **58** (1955), 545–556, 683–689.

[Lax91] P. D. Lax, *Deterministic theories of turbulence*, Frontiers in pure and applied mathematics, North-Holland, Amsterdam, 1991, pp. 179–184. MR 1110599 (92i:76058)

[Lio96] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1*, Oxford Lecture Series in Mathematics and its Applications, vol. 3, The Clarendon Press Oxford University Press, New York, 1996, Incompressible models, Oxford Science Publications. MR MR1422251 (98b:76001)

[MS01] S. Müller and M. A. Sychev, *Optimal existence theorems for nonhomogeneous differential inclusions*, J. Funct. Anal. **181** (2001), no. 2, 447–475. MR 1821703 (2002c:35281)

[MŠ03] S. Müller and V. Šverák, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. of Math. (2) **157** (2003), no. 3, 715–742. MR MR1983780 (2005i:35028)

[Nas54] J. Nash, *C^1 isometric imbeddings*, Ann. Math. **60** (1954), 383–396.

[Nas56] ———, *The imbedding problem for Riemannian manifolds*, Ann. Math. **63** (1956), 20–63.

[Ons49] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento (9) **6** (1949), no. Supplemento, 2 (Convegno Internazionale di Meccanica Statistica), 279–287. MR 0036116 (12,60f)

[Oxt80] J. C. Oxtoby, *Measure and category*, second ed., Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York, 1980, A survey of the analogies between topological and measure spaces. MR MR584443 (81j:28003)

[Pog51] A.V. Pogorelov, *The rigidity of general convex surfaces*, Doklady Acad. Nauk SSSR **79** (1951), 739–742.

[Pog73] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, American Mathematical Society, Providence, R.I., 1973, Translations of Mathematical Monographs, Vol. 35.

[Rob03] R. Robert, *Statistical hydrodynamics (Onsager revisited)*, Handbook of mathematical fluid dynamics, Vol. II, North-Holland, Amsterdam, 2003, pp. 1–54. MR 1983588 (2005a:76004)

[Sch93] V. Scheffer, *An inviscid flow with compact support in space-time*, J. Geom. Anal. **3** (1993), no. 4, 343–401. MR MR1231007 (94h:35215)

[Shn97] A. Shnirelman, *On the nonuniqueness of weak solution of the Euler equation*, Comm. Pure Appl. Math. **50** (1997), no. 12, 1261–1286. MR MR1476315 (98j:35149)

[Shn00] ———, *Weak solutions with decreasing energy of incompressible Euler equations*, Comm. Math. Phys. **210** (2000), no. 3, 541–603. MR MR1777341 (2002g:76009)

[Shv11] R. Shvydkoy, *Convex integration for a class of active scalar equations*, J. Amer. Math. Soc. **24** (2011), no. 4, 1159–1174. MR 2813340

[Spi79] M. Spivak, *A comprehensive introduction to differential geometry. Vol. V. 2nd ed.*, Berkeley: Publish or Perish, Inc., 1979 (English).

[SW12] L. Székelyhidi, Jr. and E. Wiedemann, *Young measures generated by ideal incompressible fluid flows*, Arch. Rat. Mech. Anal. **206** (2012), no. 1, 333–366.

[Szé11] L. Székelyhidi, *Weak solutions to the incompressible Euler equations with vortex sheet initial data*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 19-20, 1063–1066. MR 2842999

[Szé12] L. Székelyhidi, Jr., *Relaxation of the incompressible porous medium equation*, Ann. Sci. l'ENS **45** (2012), no. 3, 491–509.

[Tar79] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math., vol. 39, Pitman, Boston, Mass., 1979, pp. 136–212. MR MR584398 (81m:35014)

[Tar83] ———, *The compensated compactness method applied to systems of conservation laws*, Systems of nonlinear partial differential equations (Oxford, 1982), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 111, Reidel, Dordrecht, 1983, pp. 263–285. MR MR725524 (85e:35079)

[Wie11] E. Wiedemann, *Existence of weak solutions for the incompressible Euler equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011), no. 5, 727–730.

Institut für Mathematik, Universität Leipzig, D-04103 Leipzig
E-mail address: `laszlo.szekelyhidi@math.uni-leipzig.de`